Symplectic embeddings of ellipsoids in dimension greater than four

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Abstract

We study symplectic embeddings of ellipsoids into balls. In the main construction, we show that a given embedding of $2m-$dimensional ellipsoids can be suspended to embeddings of ellipsoids in any higher dimension. In dimension 6, if the ratio of the areas of any two axes is sufficiently large then the ellipsoid is flexible in the sense that it fully fills a ball. We also show that the same property holds in all dimensions for sufficiently thin ellipsoids $E(1,\dots,a)$. A consequence of our study is that in arbitrary dimension a ball can be fully filled by any sufficiently large number of identical smaller balls, thus generalizing a result of Biran valid in dimension 4.

1 Introduction

Let $E(a_1,\dots,a_n)\subset\mathbb{R}^{2n}$ be the ellipsoid

$$E(a_1,\dots,a_n)=\left\{\sum_{i=1}^{n}\frac{\pi(x_i^2+y_i^2)}{a_i}\leq 1\right\}.$$

Ellipsoids inherit a symplectic structure from the standard form $\omega_0=\sum_{i=1}^n dx_i\wedge dy_i$ on $\mathbb{R}^{2n}$. Then, in our notation, the ball of capacity c is written

$$B^{2n}(c)=E(c,\dots,c).$$

Let us also write $\lambda E(a_1,\dots,a_n)$ and $\lambda B(c)$ for the ellipsoid $E(\lambda a_1,\dots,\lambda a_n)$ and ball $B(\lambda c)$ respectively. Throughout the paper the notation

$$E(a_1,\dots,a_n)\hookrightarrow E(b_1,\dots,b_n)$$

will mean that for all $\lambda > 1$ there exists a symplectic embedding $E(a_1, \dots, a_n) \hookrightarrow \lambda E(b_1, \dots, b_n)$.

We are interested in the problem of determining when there exists a symplectic embedding from a given ellipsoid into (an arbitrarily small neighborhood of) the ball of capacity c.

This problem has been completely solved when $n = 2$, that is, in dimension 4, in the sense that the function

$$g(a) := \inf\{c | E(1, a) \hookrightarrow B^4(c)\}$$

is described in the paper [21], Theorem 1.1.2, see our Proposition 2.6.

Here we begin a systematic study of the corresponding functions in higher dimensions. The main construction that we introduce allows us to extend known results on embeddings in low dimension to higher dimensional embeddings:

Proposition 1.1. *Suppose that* $E(a_1, \dots, a_m) \hookrightarrow E(a'_1, \dots, a'_m)$. *Then also* $E(a_1, \dots, a_m, a_{m+1}, \dots, a_n) \hookrightarrow E(a'_1, \dots, a'_m, a_{m+1}, \dots, a_n)$ *for any values* $a_{m+1}, \dots, a_n$.

We will focus especially on dimension 6, where the problem is to describe the function of two variables

$$f(a, b) := \inf\left\{c | E(1, a, b) \hookrightarrow B^6(c)\right\}.$$

Note that by symmetry and rescaling we may assume that $1 \le a \le b$.

We are able to describe f completely in particular when $a^2 + b^2 \le 4$ and also when $a^2 + b^2 \ge 1.41 \times 10^{101}$. In other words we have optimal embedding results when the ellipsoid is either relatively close to a ball or in the other extreme when it is, up to scale, contained in a relatively small neighborhood of a 4-dimensional ellipsoid. The results for a, b small are contained in the sequence of Lemmas 2.7, 2.8, 2.9, 2.11. More known values of $f(a, b)$ are illustrated in Figure 1. The result for a or b large is perhaps the main result of our paper and can be stated as follows.

Theorem 1.2. *If* $a^2 + b^2 \ge 1.41 \times 10^{101}$ *then* $E(1, a, b) \hookrightarrow B((ab)^{1/3})$.

This means that in the given range we have volume filling embeddings, that is, the only obstruction to embedding this class of ellipsoids into a ball comes from their volumes. In dimension 4 the analogous result is that $E(1, a) \hookrightarrow B\left(\sqrt{a}\right)$ provided that $a > 8\frac{1}{36}$.

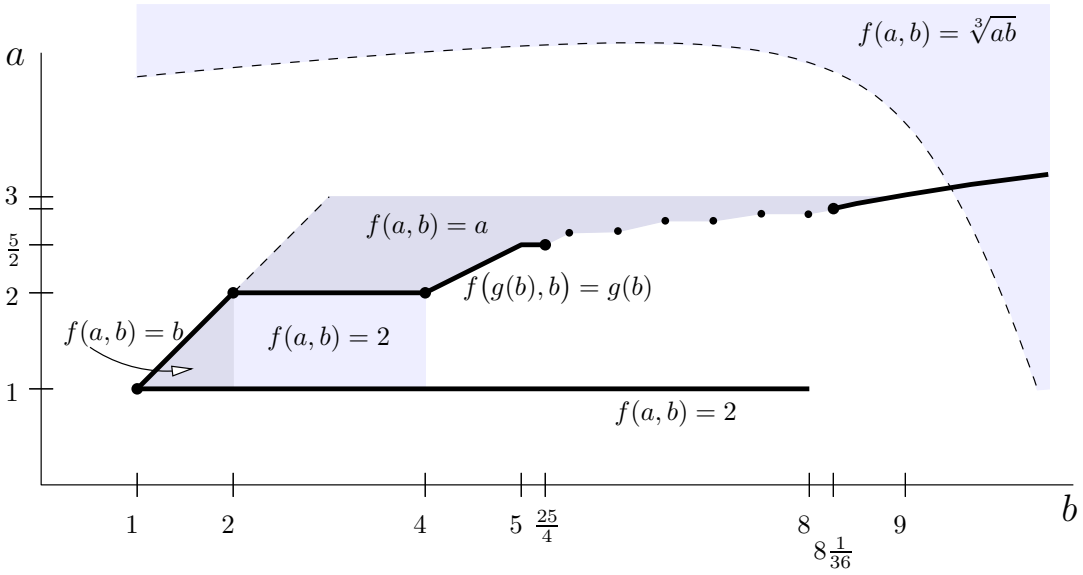

Figure 1: Known values of $f(a,b)$.

A consequence of Theorem 1.2 is a full packing (or filling) result for higher dimensional balls. Let $\sqcup_k B(c)$ be the disjoint union of k standard $2n$-dimensional balls of radius r and capacity $c = \pi r^2$.

Theorem 1.3. *For any natural number $n \geq 3$ there exists a number M_n such that for all $k \geq M_n$,*

$$\sqcup_k B\left(\tfrac{1}{k^{1/n}}\right) \hookrightarrow B^{2n}(1).$$

In other words, the round ball can be fully filled by a disjoint union of any number $k \geq M_n$ of identical balls. For the definition of M_n see definition 3.6.

The k^{th} packing number of a compact, $2n$-dimensional, symplectic manifold (M,ω) is

$$p_k(M,\omega) = \frac{\sup_c \operatorname{Vol}(\sqcup_k B(c))}{\operatorname{Vol}(M,\omega)},$$

where the supremum is taken over all c for which there exist a symplectic embedding of $\sqcup_k B(c)$ into (M,ω). Naturally, $p_k(M,\omega) \leq 1$. When $p_k(M,\omega) = 1$ we say that (M,ω) admits a full packing by k balls, otherwise we say that there is a packing obstruction.

Although no general tools are known to compute those invariants for arbitrary symplectic manifolds, some results can be derived from complex algebraic geometry using the theory of J-holomorphic curves. A first result that follows from M. Gromov [10] is that $p_i(\mathbb{CP}^n) < 1$ for any $1 < i < 2^n$. In [20], D. McDuff and L. Polterovich computed $p_i(\mathbb{CP}^2)$, for $i \leq 9$. They also proved that $p_i(\mathbb{CP}^n) = 1$ whenever $i = k^n$ and that $\lim_{i\to\infty} p_i(M,\omega) = 1$ for any compact symplectic manifold. Such results led to the natural question of whether the sequence $p_i(M,\omega)$ is eventually stable, that is, whether there is a number $N_{\text{stab}}(M,\omega)$ such that $p_i(M,\omega) = 1$ for all $i \geq N_{\text{stab}}(M,\omega)$. To date, this remains an interesting open question (see [3] and [5] for a complete discussion). When M is four dimensional, results of D. McDuff [16] and Lalonde -McDuff [12] regarding the structure of symplectic ruled surfaces and introducing inflation techniques then opened the way to a thorough study of the packing numbers. This study was done by P. Biran in a sequence of papers [1], [2] which answered the stability question positively in the cases of closed symplectic 4-manifolds whose symplectic forms (after rescaling) are in rational cohomology classes. His techniques allowed him to obtain upper and lower bounds for $N_{\text{stab}}(M^4,\omega)$ which can be explicitly computed in some cases. In particular, he showed that $N_{\text{stab}}(\mathbb{CP}^2) \leq 9$ which, in view of McDuff and Polterovich's results, is sharp. Although suspected to be true, until now there are no results in the literature proving packing stability for a symplectic manifold of dimension larger than 4. Theorem 1.3 above shows that balls admit full packings by a sufficiently large number of disjoint identical balls. As the affine part of $\mathbb{CP}^n$ is a ball, in this language Theorem 1.3 gives the following.

Theorem 1.4. *Consider* $(\mathbb{CP}^n,\omega)$ *with the symplectic form induced by the Fubini–Study metric. Then* $\mathbb{CP}^n$ *has packing stability, indeed*

$$p_i(\mathbb{CP}^n) = 1 \text{ when } i \geq M_n \tag{1}$$

Probably the bound M_n is not optimal, we briefly discuss this in Remark 4.4.

Outline of the paper.

In Section 2 we describe our basic embedding construction in Proposition 1.1 and apply it to deduce various values of $f(a,b)$, in particular for a, b sufficiently small. We remark that E. Opshtein [22] has also given a construction for embedding ellipsoids. The focus of his work is perhaps embeddings

into closed manifolds, but there is still some overlap with our own results. We mention this also in section 2.

Section 3 gives the proof of Theorem 1.2 and also constructs some volume filling embeddings of balls by ellipsoids in any dimension.

Theorem 1.3 is proven in Section 4, by combining the volume filling results of Section 3 with a construction from toric geometry.

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2 Embedding ellipsoids

2.1 The construction

Here we give the proof of our basic embedding construction. Before giving a formal proof of Proposition 1.1 we outline the general idea. Identifying each $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, let $H_s : \mathbb{C}^m \times [0,1] \to \mathbb{R}$ be a 1-parameter family of possibly time-dependent Hamiltonian functions on $\mathbb{C}^m$ and set $H(z,s,t) = H_s(z,t)$, so $H : \mathbb{C}^m \times \mathbb{R} \times [0,1] \to \mathbb{R}$. Then for any function $f : \mathbb{C}^{n-m} \to \mathbb{R}$, the function $H(z,f,t)$ can be thought of as a Hamiltonian G on $\mathbb{C}^n$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^{n-m}$ be the projection on the last $n-m$ complex coordinates, then $F = f \circ \pi : \mathbb{C}^n \to \mathbb{R}$ is an integral of the motion of G. Indeed, at all times t, we have $\{G,F\} = dG(X_F) = \frac{\partial H}{\partial s} df(X_f) = 0$. Therefore the flow of G preserves the level sets of F, and restricted to a level $\Sigma_c = \{F = c\} = \mathbb{C}^m \times \{f = c\}$ the first m (complex) components of the flow are exactly those of the Hamiltonian H_c. In other words, let Φ be the time 1 flow of G and ϕ_c be the time 1 flow of H_c. Then if $(z,w) \in \mathbb{C}^m \times \mathbb{C}^{n-m} = \mathbb{C}^n$ has $F(z,w) = f(w) = c$, we have $\Phi(z,w) = (\phi_c(z), w')$ where $f(w') = c$. Thus a domain $D \subset \mathbb{C}^n$ whose fibers $\pi^{-1}(w) \cap D = D_w = D_{f(w)}$ depend only on $f(w)$ will be mapped under Φ to a domain D' with fibers $\pi^{-1}(w) \cap D' = \phi_{f(w)}(D_{f(w)})$.

Returning to Proposition 1.1, and still identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we can write

$$E(a_1, \ldots, a_n) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{\pi|z_1|^2}{a_1} + \ldots + \frac{\pi|z_n|^2}{a_n} \leq 1 \right\}.$$

In the notation above, the fibers of $E(a_1, \ldots, a_n)$ over $\mathbb{C}^{n-m}$ are ellipsoids $rE(a_1, \ldots, a_m)$ where $r = 1 - \frac{\pi|z_{m+1}|^2}{a_{m+1}} - \ldots - \frac{\pi|z_n|^2}{a_n}$. Roughly speaking, we will apply our general idea to a 1-parameter family of Hamiltonian functions H_r whose corresponding flows map $rE(a_1, \ldots, a_m)$ into $rE(a'_1, \ldots, a'_m)$.

Proof of Proposition 1.1 Fixing a $\lambda > 1$, it is required to show that there exists a symplectic embedding $E(a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) \to \lambda E(a'_1, \ldots, a'_m, a_{m+1}, \ldots, a_n)$.

By hypothesis, we have $E(a_1, \ldots, a_m) \hookrightarrow E(a'_1, \ldots, a'_m)$. By the Extension after Restriction Principle, see [23], page 7, our hypothesis implies that there exists a $\mu > 1$ and a Hamiltonian diffeomorphism mapping $\mu E(a_1, \ldots, a_m) \to \lambda E(a'_1, \ldots, a'_m)$. Suppose that this Hamiltonian diffeomorphism is the time 1 flow corresponding to a Hamiltonian function $H : \mathbb{C}^m \times [0, 1] \to \mathbb{R}$.

Observe that for any $r > 0$ the Hamiltonian H_r defined by $H_r(z, t) = rH(\frac{z}{\sqrt{r}}, t)$ for $z \in \mathbb{C}^m$ generates a flow with time 1 map taking $r\mu E(a_1, \ldots, a_m) \to r\lambda E(a'_1, \ldots, a'_m)$. (For this, recall that in our notation the map $(z_1, \ldots, z_m) \mapsto (\sqrt{r}z_1, \ldots, \sqrt{r}z_m)$ takes $E(a_1, \ldots, a_m)$ onto $rE(a_1, \ldots, a_m)$.)

For $z \in \mathbb{C}^n$, let

$$r(z) = r(z_{m+1}, \ldots, z_n) = 1 - \frac{\pi|z_{m+1}|^2}{\mu a_{m+1}} - \ldots - \frac{\pi|z_n|^2}{\mu a_n}.$$

Define $K(z_1, \ldots, z_n, t) = H_{r(z_{m+1}, \ldots, z_n)}(z_1, \ldots, z_m, t)$ for z with $\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n} \leq 1$ and extend the function arbitrarily to the remainder of $\mathbb{C}^n \times [0, 1]$. Note that $\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n} \leq 1$ implies that $r(z) \geq 1 - \frac{1}{\mu} > 0$ and so K is well defined. We claim that K generates a Hamiltonian diffeomorphism ϕ as required.

More precisely, we claim the following. Suppose that $z \in \mathbb{C}^n$ is such that $\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n} = k \leq 1$ and $\phi(z) = w = (w_1, \ldots, w_n)$. Then

1. $\frac{\pi|w_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|w_n|^2}{a_n} = k$;

2. $\frac{\pi|w_1|^2}{a'_1} + \ldots + \frac{\pi|w_m|^2}{a'_m} \leq \lambda(1 - k)$.

Given this, $\frac{\pi|w_1|^2}{a'_1} + \ldots + \frac{\pi|w_n|^2}{a_n} \leq \lambda(1 - k) + k \leq \lambda$ and so $\phi(z) = w \in \lambda E(a'_1, \ldots, a'_m, a_{m+1}, \ldots, a_n)$ as required.

Statement (1.) follows because on the region $\{\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n} \leq 1\}$ the Hamiltonian flow of K preserves r and hence $\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n}$.

For statement (2.) note that if $\frac{\pi|z_{m+1}|^2}{a_{m+1}} + \ldots + \frac{\pi|z_n|^2}{a_n} = k$ then the partial derivatives of our Hamiltonian in the $z_1, \ldots, z_m$ directions are equal to the corresponding derivatives of $H_{1-\frac{k}{\mu}}(z_1, \ldots, z_m)$.

Now, $(z_1, \ldots, z_m) \in (1-k)E(a_1, \ldots, a_m) \subset (1-\frac{k}{\mu})\mu E(a_1, \ldots, a_m)$. Thus as the flow of $H_{1-\frac{k}{\mu}}$ takes $(1-\frac{k}{\mu})\mu E(a_1, \ldots, a_m) \to (1-\frac{k}{\mu})\lambda E(a'_1, \ldots, a'_m)$ at time 1 we have $(w_1, \ldots w_m) \in (1-\frac{k}{\mu})\lambda E(a'_1, \ldots, a'_m)$. In other words, $\frac{\pi|w_1|^2}{a'_1} + \ldots + \frac{\pi|w_m|^2}{a'_m} \leq (1-\frac{k}{\mu})\lambda < \lambda(1-k)$ as claimed.

□

Remark 2.1. As mentioned in the introduction there is related work of E. Opshtein which we outline here, for convenience focusing on the case of embeddings into $\mathbb{CP}^3(c)$, complex projective space equipped with the Fubini–Study form scaled such that lines have symplectic area c. The symplectic manifold $\mathbb{CP}^3(c)$ is of special interest to us as the affine part is symplectomorphic to the 6-ball of capacity c. Opshtein observes the following.

Theorem 2.2 (Opshtein, [22])**.** *Let $\Sigma \subset \mathbb{CP}^3(c)$ be a smooth holomorphic hypersurface of degree k. Then any symplectic embedding $E(a,b) \hookrightarrow \Sigma$ extends to a symplectic embedding $E(\frac{c}{k}, a, b) \hookrightarrow \mathbb{CP}^3(c)$.*

For example, if $k = 1$ then Σ is a copy of $\mathbb{CP}^2$ which contains an embedded ball $B^4(c)$ of capacity c. Thus if $E(a,b) \hookrightarrow B(c)$ then we also find an embedding $E(c,a,b) \hookrightarrow \mathbb{CP}^3(c)$. Under the same hypotheses our Proposition 1.1 also gives an embedding $E(c,a,b) \hookrightarrow B^6(c) \subset \mathbb{CP}^3(c)$. We expect the two embeddings are symplectically isotopic.

2.2 Embedding obstructions

To check that our constructions are sharp we rely only on the volume obstruction and on the Ekeland-Hofer capacities, see [6] and [7]. The volume obstruction says the following.

Proposition 2.3 (Liouville's Theorem)**.** *If $E(a_1, \ldots, a_n) \hookrightarrow E(b_1, \ldots, b_n)$ then $a_1 \cdots a_n \leq b_1 \cdots b_n$.*

The Ekeland–Hofer capacities give an infinite sequence of numbers $c_k(E(a_1,\dots,a_n))$ associated to an ellipsoid. In our situation we can take the definition to be as follows.

Definition 2.4. $c_k(E(a_1,\dots,a_n))$ is the k^{th} number in the ordered sequence (with repetitions if necessary) of numbers in the set $\{k_1a_1,\dots,k_na_n|k_i\in\mathbb{N}\}$.

Theorem 2.5 ([6], [7])**.** *If* $E(a_1,\dots,a_n)\hookrightarrow E(b_1,\dots,b_n)$ *then* $c_k(E(a_1,\dots,a_n))\leq c_k(E(b_1,\dots,b_n))$ *for all* k*.*

Note that $c_k(\lambda E(a_1,\dots,a_n))=\lambda c_k(E(a_1,\dots,a_n))$. Therefore if $E(a_1,\dots,a_n)\hookrightarrow E(b_1,\dots,b_n)$ and $c_k(E(a_1,\dots,a_n))=c_k(E(b_1,\dots,b_n))$ for some k, we know that the embedding is optimal in the sense that there is no embedding $E(a_1,\dots,a_n)\hookrightarrow\mu E(b_1,\dots,b_n)$ for any $\mu<1$.

2.3 Some calculations in dimension 3

Here we give some optimal embeddings for ellipsoids $E(1,a,b)$ when a and b are relatively small. We recall the definitions of the functions f and g from the introduction.

$$f(a,b):=\inf\big\{c|E(1,a,b)\hookrightarrow B^6(c)\big\},$$
$$g(a):=\inf\big\{c|E(1,a)\hookrightarrow B^4(c)\big\}.$$

The function g is completely determined in the paper [21]. In this paper we apply only a small amount of information about g which is summarized in the following proposition.

Proposition 2.6 (McDuff–Schlenk, [21, Theorem 1.1.2])**.** *If* $b=4$ *or* $b>8\frac{1}{36}$*, then* $g(b)=\sqrt{b}$*.*

We always assume without loss of generality that $1\leq a\leq b$.

Lemma 2.7. $f(g(b),b)=g(b)$*.*

Proof. We observe that

$$E(1,g(b),b)\cong E(g(b),1,b)\hookrightarrow E(g(b),g(b),g(b))=B(g(b)),$$

where the arrow follows from Proposition 1.1 and the definition of g. Thus $f(g(b),b)\leq g(b)$.

Now, it follows from Proposition 2.6 that $g(b) = \sqrt{b}$ whenever $b \geq 9$. Thus the embedding is optimal in this case by Proposition 2.3.

If $b \leq 9$ then $g(b) \leq 3$. In the first case suppose that $1 \leq g(b) \leq 2$. Then

$$c_2(E(1, g(b), b)) = g(b) = c_2(B(g(b))).$$

In the second case, if $2 \leq g(b) \leq 3$ then

$$c_3(E(1, g(b), b)) = g(b) = c_3(B(g(b))).$$

Thus the embedding is also optimal in both of these cases by Theorem 2.5. □

Lemma 2.8. *Suppose that $a \leq 3$ and $g(b) \leq a$. Then $f(a, b) = a$.*

Proof. The embedding construction here is as in the previous proof of Lemma 2.7, and our hypothesis are such that the third Ekeland–Hofer capacity again implies that it is optimal. □

Lemma 2.9. *Suppose that $1 \leq a \leq b \leq 2$. Then $f(a, b) = b$.*

Proof. $E(1, a, b) \hookrightarrow B(b)$ simply by inclusion. But

$$c_3(E(1, a, b)) = b = c_3(B(b))$$

for our range of a, b, so the inclusion is optimal. □

Remark 2.10. It is true in any dimension that if $a_n \leq 2a_1$ then $E(a_1, \ldots, a_n) \hookrightarrow B(c)$ if and only if $c \geq a_n$. This was established in the case of $n = 2$ by Floer, Hofer and Wysocki in [8] as an application of symplectic homology. The theorem stated here is Theorem 1 of [23], where Schlenk gives a simple proof by applying the nth Ekeland-Hofer capacity as in our proof of Lemma 2.9 above. We thank the referee for reminding us of this slightly strange history.

Lemma 2.11. *Suppose that $1 \leq a \leq 2 \leq b \leq 4$. Then $f(a, b) = 2$.*

Proof.

$$E(1, a, b) \cong E(a, 1, b) \hookrightarrow E(a, 2, 2) \hookrightarrow B(2)$$

where the first arrow follows from Proposition 1.1 since $g(b) = 2$ (see [21, Figure 1.1]) and the second arrow is the inclusion.

The third Ekeland–Hofer capacity

$$c_3(E(1, a, b)) = 2 = c_3(B(2))$$

and so our construction is optimal. □

We give one final computation which will rely on the following.

Lemma 2.12. $E(1,1,8) \hookrightarrow B(2)$.

Proof. This is a particular case of Lemma 3.5 below. □

Lemma 2.13. *Suppose that* $2 \leq b \leq 8$. *Then* $f(1,b) = 2$.

Proof. $f(1,b)$ is an increasing function of b. We know that $f(1,4) = 2$ by Lemma 2.11 and Lemma 2.12 says that $f(1,8) = 2$. Thus $f(1,b)$ is in fact constant on the interval $4 \leq b \leq 8$. □

3 Volume filling embeddings

In this section we prove Theorem 3.8, a more precise version of Theorem 1.2. First, in subsection 3.1 we recall two theorems of D. McDuff on embeddings in dimension 4. The first reduces an ellipsoid embedding problem to one of embedding a disjoint union of balls into a ball, the second gives necessary conditions for embedding a disjoint union of balls. We close the subsection with two useful consequences. In subsection 3.2 we use these theorems to derive some preliminary results on embedding ellipsoids in higher dimensions. We think these are quite interesting in themselves; the main result is Proposition 3.7 which will be applied in section 4 to give our full packing result. Finally in subsection 3.3 we prove Theorem 3.8.

3.1 Four dimensional embeddings

Here we review some results of D. McDuff which allow one to translate a 4-dimensional ellipsoid embedding problem into one of a disjoint union of balls, and then give an algebraic solution for the ball embedding problem.

Proposition 3.1 (D. McDuff [17], [18])**.** *Let* e, f, c, d *be positive integers with* $e \leq f$ *and* $c \leq d$. *There exists a weight expansion* $W(e,f)$ *associated to any pair of integers such that a symplectic embedding of ellipsoids* $E(e,f) \longrightarrow E(c,d)$ *exists if and only if there exists a symplectic embedding of balls*

$$\big(\sqcup B(W(e,f))\big) \cup \big(\sqcup B(W(d-c,d))\big) \hookrightarrow B(d). \tag{2}$$

Let us explain the weight sequences $W(e,f)$, as they are defined for instance in [18].

$$W(e,f) = (X_0^{\times l_0}, X_1^{\times l_1}, \dots, X_K^{\times l_K}) \tag{3}$$

where the multiplicities l_i are the entries in the continued fraction expansion

$$\frac{f}{e} = [l_0; l_1, \dots, l_k] = l_0 + \cfrac{1}{l_1 + \cfrac{1}{\ddots \frac{1}{l_k}}}$$

and the entries X_i are defined inductively as follows:

$$X_{-1} = f, X_0 = e, X_{i+1} = X_{i-1} - l_i X_i. \tag{4}$$

We use the notation

$$\sqcup B(W(e,f)) := \left(\sqcup_{i=1}^{l_0} B(X_0) \right) \cup \left(\sqcup_{i=1}^{l_1} B(X_1) \right) \cup \ldots \cup \left(\sqcup_{i=1}^{l_K} B(X_K) \right). \tag{5}$$

In the paper [20], McDuff and Polterovich equated the problem of embedding M disjoint balls with understanding the symplectic cone of the M-fold blow-up X_M of $\mathbb{CP}^2$. The structure of the cone was intensely studied by Biran [1] and later Li-Liu [13] and Li-Li [14]. To explain their results, denote by $L, E_1, \dots, E_M \in H_2(X_m, \mathbb{Z})$ the homology classes of the line and the exceptional divisors, by $-K := 3L - \sum_i E_i$ the anticanonical class, and by $l, e_1, \dots, e_M$ their Poincaré duals. With respect to the basis $L, -E_1, \dots, -E_M$ we represent homology classes by $(M+1)$-tuples $(d, \overline{m})$. We fix a symplectic form ω_M on X_M obtained by symplectically blowing up $(\mathbb{CP}^2, \omega)$ with the standard Fubini–Study form ω. Define $\mathcal{C}_M$ to be the set of all cohomology classes in $H^2(X_M)$ that can be represented by symplectic forms whose first Chern classes are Poincaré dual to $-K$. Next define the exceptional cone to be the set of homology classes $\mathcal{E}_M \subset H_2(X_M, \mathbb{Z})$,

$$\mathcal{E}_M := \{E \mid E \cdot E = -1 \text{ and } E \text{ is represented by an embedded } \omega_M\text{-symplectic sphere}\}.$$

Since the classes E in the exceptional cone have nontrivial Gromov invariants, the definition of the exceptional cone is independent of the choice of ω_M. In [14] Li-Li showed that

$$\mathcal{C}_M = \left\{\alpha \in H^2(X_M) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \text{ for all } E \in \mathcal{E}_M\right\}.$$

Given these definitions, the work of McDuff-Polterovich, Biran, Li-Liu and Li-Li, gives the following criteria for embedding disjoint unions of balls.

Proposition 3.2 (See [20], [1], [14], [15], [13])**.** *A symplectic ball embedding* $\sqcup_{i=1\ldots,M} B^4(w_i) \hookrightarrow B^4(\mu)$ *exists if, and only if,* $\mu l - \sum_{i=1}^{M} w_i e_i \in \mathcal{C}_M$*. This is equivalent to the following two conditions.*

1. $\mu^2 \geq \sum_{i=1}^{M} w_i^2$
2. *For any* $(d', \overline{m}) \in \mathcal{E}_M$*, we have:*

$$d'\mu \geq \sum_{i=1}^{M} m_i w_i \tag{6}$$

We close this subsection with the following two applications of these results.

Proposition 3.3. *For any* $k, x \in \mathbb{N}$*, the following embedding holds:*

$$E(1, k^{2x+1}) \hookrightarrow E(k^x, k^{x+1}). \tag{7}$$

Proof. We apply Proposition 3.1 with $e = 1, f = k^{2x+1}, c = k^x, d = k^{x+1}$. Then the equivalent embedding (2) (see the notation (5)) becomes

$$\sqcup B\big(W(1, k^{2x+1})\big) \cup \sqcup B\big(W(k^{x+1} - k^x, k^{x+1})\big) \hookrightarrow B(k^{x+1}). \tag{8}$$

Since the continuous fraction expansions are $\frac{1}{k^{2x+1}} = [0; k^{2x+1}]$ and $\frac{k^{x+1}-k^x}{k^{x+1}} = [0; 1, k-1]$, the vectors described in (4) express this last embedding as

$$\Big(\sqcup_{i=1}^{k^{2x+1}} B_i(1) \Big) \cup B(k^{x+1} - k^x) \cup \Big(\sqcup_{j=1}^{k-1} B(k^x) \Big) \hookrightarrow B(k^{x+1}). \tag{9}$$

This embedding can be seen from a toric viewpoint. Indeed, there exists a toric decomposition of the ball of capacity k^{x+1} containing an open ball of capacity $k^{x+1} - k^x$ and the preimages of $2k-1$ polytopes of capacity k^x as shown in Figure 2. For the fact that these open toric manifolds admit embeddings of an open ball of the stated capacity see [26], Proposition 5.2. But a ball of capacity k^x can be filled with k^{2x} balls of capacity 1, see [26] again, Construction 3.2, for an explicit construction. If we decompose k of our $2k-1$ such balls in this way then we get the embedding as required.

This concludes the proof of Proposition 3.3. □

Lemma 3.4. *If* $\sqrt{\frac{2}{3}} \leq \lambda \leq 1$ *and* $b \geq 9$*, then* $E(1, b) \hookrightarrow E(\lambda\sqrt{b}, \lambda^{-1}\sqrt{b})$.

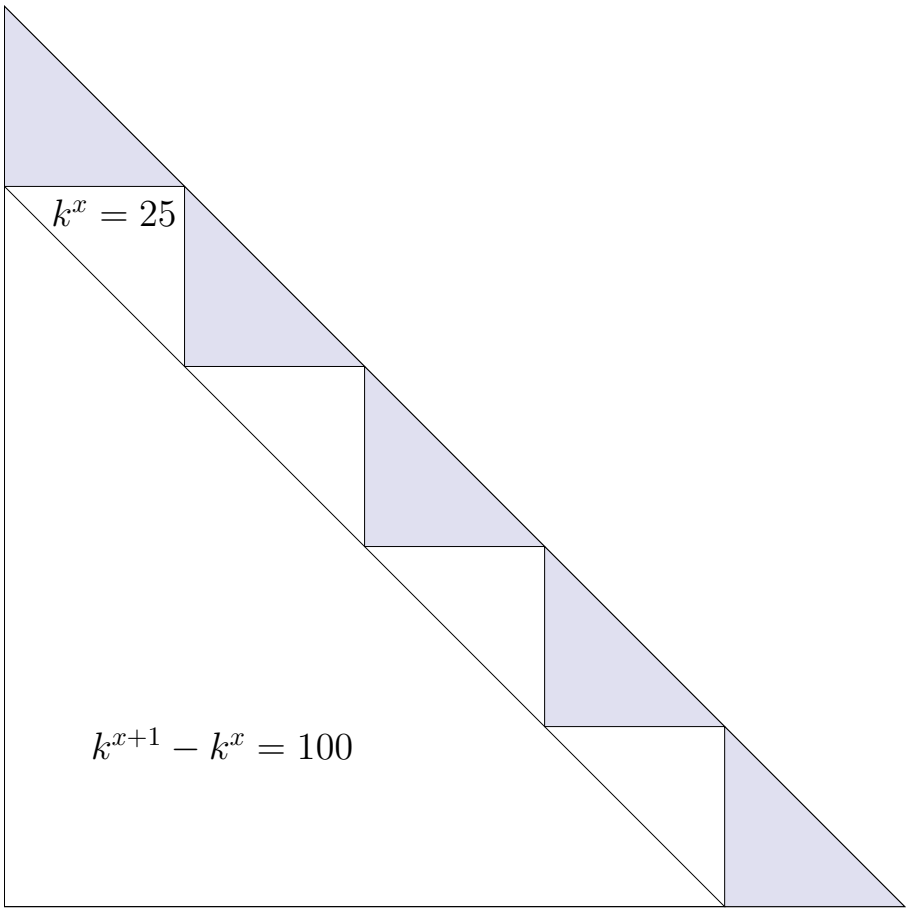

Figure 2: A toric decomposition of the ball when $k = 5, x = 2$. The same toric packing strategy applies to any k, x natural.

Proof. The proof largely follows the method used by McDuff and Schlenk in [21], Corollary 1.2.4, where they establish the case $\lambda = 1$. It is sufficient to consider the case when both $\lambda = \frac{u}{v}$ and $\sqrt{b} = \frac{p}{q}$ rational, with $p \geq 3q$, $\sqrt{\frac{2}{3}} \leq \frac{u}{v} \leq 1$.

Then we need to prove that

$$E\big(1, \tfrac{p^2}{q^2}\big) \hookrightarrow E\big(\tfrac{up}{vq}, \tfrac{vp}{uq}\big) \tag{10}$$

and the latter is equivalent to showing that there exists an embedding

$$E(uvp^2, uvq^2) \hookrightarrow E(u^2pq, v^2pq). \tag{11}$$

Using now Proposition 3.1 we see that this is equivalent to the existence of the following ball embedding:

$$\Big(\sqcup B\big(W(uvq^2, uvp^2)\big)\Big) \cup \Big(\sqcup B\big(W((v^2-u^2)pq, v^2pq)\big)\Big) \hookrightarrow B(v^2pq). \tag{12}$$

We will use the criteria from Proposition 3.2 to show that the embedding (12) does indeed exist.

The first condition is clearly satisfied because our embeddings are volume preserving. To verify the second, note that since $(d', \overline{m}) \in \mathcal{E}_M$ we have $3d' - 1 = \sum_{i=1}^{M} m_i$.

Let $\sqcup B\big(W(uvq^2, uvp^2)\big) = \sqcup_{i=1}^{M_1} B(w_i)$ and $\sqcup B\big(W((v^2-u^2)pq, v^2pq)\big) = \sqcup_{i=M_1+1}^{M} B(w_i')$.

We have

$$\sum_{i=1}^{M_1} m_i w_i + \sum_{i=M_1+1}^{M} m_i w_i' \leq \sum_{i=1}^{M_1} uvq^2 m_i + \sum_{i=M_1+1}^{M} \big((v^2-u^2)pq\big) m_i \tag{13}$$

since each $w_i \leq uvq^2$ and each $w_i' \leq (v^2-u^2)pq$. Therefore

$$\sum_{i=1}^{M_1} m_i w_i + \sum_{i=M_1+1}^{M} m_i w_i' \leq \max\big\{uvq^2, (v^2-u^2)pq)\big\} \sum_{i=1}^{M} m_i = \max\big\{uvq^2, (v^2-u^2)pq)\big\}(3d'-1). \tag{14}$$

Thus, to show the inequality (6), it is sufficient to verify that both of the following inequalities hold:

$$uvq^2(3d'-1) \leq v^2pqd'; \tag{15}$$

$$(v^2 - u^2)pq(3d' - 1) \leq v^2 pqd'. \tag{16}$$

The first inequality is guaranteed if $3uvq^2 \leq v^2 pq$, or $\frac{u}{v} \leq \frac{1}{3}\frac{p}{q}$. This is certainly true as $\lambda \leq 1$ and $\sqrt{b} \geq 3$.

The second is guaranteed if $3(v^2 - u^2)pq \leq v^2 pq$, or, $\frac{u^2}{v^2} \geq \frac{2}{3}$. This is true as well since we assumed $\lambda^2 \geq \frac{2}{3}$.

□

3.2 Applications to higher dimensions

We start this section by applying Proposition 3.3 to obtain embeddings in higher dimensions.

For brevity, we will use the notation $E(a_1, a_2, \ldots, a_i^{\times m}, a_{i+m}, \ldots a_n)$ if an entry a_i is repeated m times.

Lemma 3.5. *For any $k, n \in \mathbb{N}$,*

$$E(1^{\times(n-1)}, k^n) \hookrightarrow B(k). \tag{17}$$

Proof. We fix k and proceed by induction on n. First, (17) is obviously true when $n = 1$, while the case $n = 2$, first proved in [17], can be easily seen as part of McDuff–Schlenk's results on the values of $g(b)$ from Proposition 2.6.

Let $n \geq 3$. If n is odd, say $n = 2m + 1$, we have

$$E(1^{\times(n-1)}, k^n) \hookrightarrow E(1^{\times(n-2)}, k^m, k^{m+1}) = E(1^{\times(m-1)}, k^m, 1^{\times m}, k^{m+1}),$$

where the first embedding is from Proposition 3.3. By the induction hypothesis and Proposition 1.1, the final ellipsoid embeds into

$$E(k^{\times m}, k^{\times(m+1)}) = E(k^{\times n}).$$

On the other hand, if n is even, say $n = 2m$, then

$$E(1^{\times(n-1)}, k^n) = E(1^{\times m}, 1^{\times(m-1)}, \left(k^2\right)^m),$$

which by the induction hypothesis (with k^2, m instead of k, n) and Proposition 1.1 again embeds into

$$E(1^{\times m}, \left(k^2\right)^{\times m}).$$

But by using Proposition 1.1 repeatedly and the fact that $E(1, k^2) \hookrightarrow E(k, k)$ (see again Proposition 2.6) we can split every instance of k^2 into two copies of k, which ends the proof.

□

Next, Lemma 3.4 will be used in order to prove a similar result to Lemma 3.5, but replacing k^n with a real number.

Definition 3.6. Define the sequence M_i, $i \geq 2$ inductively as follows: $M_2 = 8\frac{1}{36}$, $M_n = \max(M_{n-1}^2, \beta_n)$, where

$$\beta_n = \left(\frac{\sqrt[2n-2]{3}}{\sqrt[2n-2]{3} - \sqrt[2n-2]{2}} \right)^{\frac{2n(n-1)}{n-2}} \tag{18}$$

We are now in a position to state the main result of this subsection.

Proposition 3.7. *If b is any real number larger or equal to M_n, then $E(1^{\times(n-1)}, b) \hookrightarrow B(b^{\frac{1}{n}})$.*

Proof. We will proceed by induction with respect to the dimension n. The result holds when $n = 2$ by [21], see Proposition 2.6. Assume that the result holds for ellipsoids of complex dimension less or equal than $n - 1$.

We set

$$\lambda = \left(\frac{\left\lfloor b^{\frac{n-2}{2n(n-1)}} \right\rfloor}{b^{\frac{n-2}{2n(n-1)}}} \right)^{n-1}.$$

Then since $b \geq \beta_n$ we observe that

$$\sqrt{\frac{2}{3}} \leq \left(\frac{b^{\frac{n-2}{2n(n-1)}} - 1}{b^{\frac{n-2}{2n(n-1)}}} \right)^{n-1} \leq \lambda \leq 1.$$

By Lemma 3.4 and Proposition 1.1 we have the embedding

$$E(1^{\times(n-1)}, b) \hookrightarrow E(1^{\times(n-2)}, \lambda^{-1}\sqrt{b}, \lambda\sqrt{b}) \tag{19}$$

Meanwhile, since $\lambda^{-1}\sqrt{b} \geq M_{n-1}$, by the induction hypothesis we have the embedding

$$E\left(1^{\times(n-2)}, \lambda^{-1}\sqrt{b}, \lambda\sqrt{b}\right) \hookrightarrow E\left(\left((\lambda^{-1}\sqrt{b})^{\frac{1}{n-1}}\right)^{\times(n-1)}, \lambda\sqrt{b}\right) = \left(\lambda^{-1}\sqrt{b}\right)^{\frac{1}{n-1}} E\left(1^{\times(n-1)}, z\right), \tag{20}$$

where

$$z = \frac{\lambda\sqrt{b}}{(\lambda^{-1}\sqrt{b})^{\frac{1}{n-1}}} = \left(\lambda^{\frac{n}{n-1}}\right) \cdot \left(b^{\frac{n-2}{2n-2}}\right) = \left\lfloor b^{\frac{n-2}{2n(n-1)}} \right\rfloor^n.$$

Thus our result follows from the corresponding result for integers, Lemma 3.5.

□

3.3 Proof of Theorem 1.2

Here we show the following.

Theorem 3.8. *Suppose that*

$$b > (M_3)^4 a^2$$

or

$$a \geq 8\tfrac{1}{36} \text{ and } b > (M_3)^2.$$

Then $E(1,a,b) \hookrightarrow B\left((ab)^{\frac{1}{3}}\right)$.

Remark 3.9. As $1 \leq a \leq b$, the hypotheses of Theorem 3.8 are automatically satisfied if

$$a^2 + b^2 \geq \left(8\tfrac{1}{36}\right)^2 \left(1 + \left(8\tfrac{1}{36}\right)^2 \left(\frac{\sqrt[4]{3}}{\sqrt[4]{3} - \sqrt[4]{2}}\right)^{96}\right) > 1.41 \times 10^{101}.$$

Proof. First suppose that $b > (M_3)^4 a^2$. We note that this automatically implies that $b > 8\frac{1}{36}$ and $\frac{\sqrt{b}}{a} > 8\frac{1}{36}$. Then we have

$$E(1,a,b) \hookrightarrow E\left(a, \sqrt{b}, \sqrt{b}\right) =$$
$$aE\left(1, \tfrac{\sqrt{b}}{a}, \tfrac{\sqrt{b}}{a}\right) \hookrightarrow aE\left(\tfrac{b^{1/4}}{a^{1/2}}, \tfrac{b^{1/4}}{a^{1/2}}, \tfrac{\sqrt{b}}{a}\right) =$$
$$a^{\frac{1}{2}} b^{\frac{1}{4}} E\left(1, 1, \tfrac{b^{1/4}}{a^{1/2}}\right) \hookrightarrow a^{\frac{1}{2}} b^{\frac{1}{4}} B\left(\tfrac{b^{1/12}}{a^{1/6}}\right) = B((ab)^{1/3}).$$

The first embedding is possible since $b > 8\frac{1}{36}$, the second since $\frac{\sqrt{b}}{a} > 8\frac{1}{36}$, and the third by Proposition 3.7 since $\frac{b^{1/4}}{a^{1/2}} > M_3$.

Next we assume that $a \geq 8\frac{1}{36}$ and $b > (M_3)^2$. Then we have embeddings

$$E(1,a,b) \hookrightarrow E(\sqrt{a},\sqrt{a},b) =$$
$$\sqrt{a}E\left(1,1,\tfrac{b}{\sqrt{a}}\right) \hookrightarrow \sqrt{a}B\left(\tfrac{b^{1/3}}{a^{1/6}}\right) = B((ab)^{1/3}).$$

Now the first embedding relies on $a \geq 8\frac{1}{36}$ and the second exists by Proposition 3.7 again, since $\frac{b}{\sqrt{a}} > \sqrt{b} > M_3$.

□

4 Packing stability in $\mathbb{CP}^n$

In this section we prove Theorem 1.3. It relies on the following generalization to higher dimensions of a 4-dimensional polytope decomposition introduced by D. McDuff in [17].

Lemma 4.1. *For any $k \in \mathbb{N}$,*

$$\sqcup_k B(1) \hookrightarrow E\big(1^{\times(n-1)}, k\big).$$

That is, for any $\rho < 1$ the disjoint union of k balls of capacity ρ can be symplectically embedded in the ellipsoid $E(1^{\times(n-1)}, k)$.

Remark 4.2. Lemma 2.6 of [17] proves a stronger result in dimension 4, namely that there exists a symplectic embedding $\sqcup_k \mathring{B}^4(1) \to \mathring{E}\big(1,k\big)$ from the interiors of the balls to the interior of the ellipsoid. As far as we know, it is unknown if such an embedding exists in higher dimension. Nevertheless, we give a proof of Lemma 4.1 following this lemma.

There is an alternative approach following [26], Chapters 5 and 6, see also [23], section 9.4. The idea is to approximate both the ellipsoid and the balls by Lagrangian products. Once this is done correctly it is easy to see that at least an arbitrarily large compact subset of the k open balls can be symplectically embedded in the ellipsoid.

Proof of Lemma 4.1. There is a natural action of the torus T^n on the open ellipsoid $\mathring{E}\big(1^{\times(n-1)}, k\big) \subset \mathbb{C}^n$ given by rotation in each of the n complex coordinates. Let $\{\boldsymbol{e}_j\}$ denote the standard basis in $\mathbb{R}^n$. Then the moment polytope Δ of the corresponding moment map is the convex hull of the set

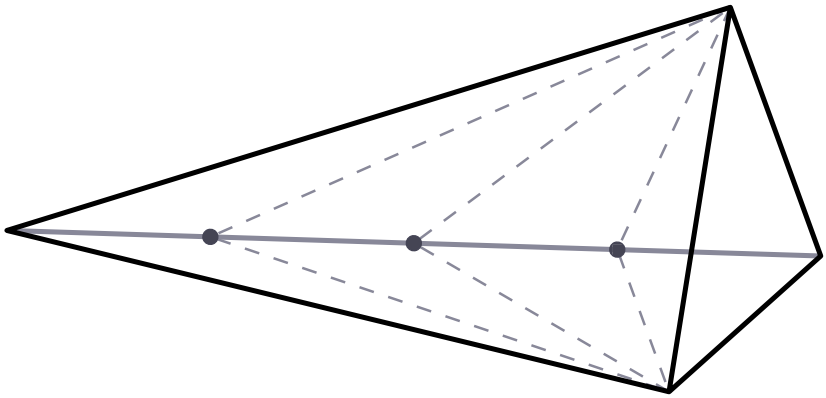

Figure 3: Division of the polytope Δ.

$\{0, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n-1}, k\boldsymbol{e}_n\}$ minus the diagonal; that is, the n-dimensional polytope of vectors $p = (x_1, \ldots, x_n)$ that satisfy

$$x_j \geq 0, \text{ and } x_1 + \ldots + x_{n-1} + \tfrac{x_n}{k} < 1.$$

Inside Δ we can find k sub-polytopes Δ_j defined by taking the interior of the convex hull of $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}, (j-1)k\boldsymbol{e}_n, jk\boldsymbol{e}_n\}$ for $j = 1 \ldots k$, see Figure 3. The affine map $\Theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that fixes the first $n-1$ coordinates and takes the n^{th} coordinate x_n to $x_n - \big[k - (x_1 + \ldots + x_{n-1})\big]$ has integer coefficients and takes each Δ_j onto the previous polytope Δ_{j-1} so that each Δ_j maps onto Δ_1 by the integral affine map Θ^{j-1}.

Now, one can see that Δ_1 is the moment polytope of the open subset

$$U = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq 0, \pi|z_1|^2 + \ldots + \pi|z_n|^2 < 1\} \subset B(1) \subset \mathbb{C}P^n.$$

This contains an embedded symplectic open ball of capacity ρ for any $\rho < 1$, for this, one can apply a product map such as Ψ^ρ defined in [26], page 420.

From the above, all other Δ_j must also admit an embedding of a ball of the same capacity. Thus, we have found an embedding

$$\sqcup_k B(\rho) \to E\big(1^{\times(n-1)}, k\big)$$

for any $\rho < 1$, as required.

□

Remark 4.3. The result above enables us to find embeddings of disjoint balls into a space whenever ellipsoid embeddings of the specified type are available. Theorem 1.1 in [17] also gives a converse, that an ellipsoid embedding exists whenever the corresponding ball embedding does. This result uses an orbifold blow up as in L. Godinho [9], see also McDuff, [19, Section 2.2], then the theory of holomorphic curves and the rational blow down constructions due to Symington, [25]. No such technology is yet available in higher dimension.

Proof of Theorem 1.3. Rescaling, it is required to show that for all natural numbers $k \geq M_n$ there exists an embedding

$$\sqcup_k B(1) \hookrightarrow B^{2n}(k^{1/n}).$$

By Proposition 3.7 there exists an embedding

$$E\big(1^{\times(n-1)}, k\big) \hookrightarrow B(k^{1/n}),$$

so it suffices to find an embedding

$$\sqcup_k B(1) \hookrightarrow E\big(1^{\times(n-1)}, k\big).$$

But this is Lemma 4.1. □

Remark 4.4. *Improving the bounds.* Our stability bound M_n in Theorems 1.3 and 1.4 seem far from optimal. Improving these bounds will be the topic of a future work [4]. This will follow from improving the bound M_n in Proposition 3.7. Now, according to [18], one 4-dimensional ellipsoid embeds into another if and only if there are no obstructions coming from the Embedded Contact Homology capacities defined by M. Hutchings in [11]. Therefore we have a route to proving Proposition 3.3 for real rather than just integer k by studying the behavior of the ECH capacities. Once this is done, Proposition 3.7 follows exactly as Lemma 3.5.

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