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Deformations of Whitehead Products, Symplectomorphism Groups, and Gromov–Witten Invariants

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We provide a new way of understanding the multiplicative structure of the rational homotopy groups $\pi_*(X_\lambda) \otimes \mathbb{Q}$ for a family of topological spaces, once we know enough about their additive structure. This allows us to interpret the condition of realizing as an A_k map a multiple of a map $f : S^1 \longrightarrow G$ between two topological groups in terms of the existence of a rational Whitehead product of order k. Our main example will be when the X_λ are classifying spaces of symplectomorphism groups $\mathrm{BSymp}(\Sigma_g \times S^2, \omega_\lambda)$ where ω_λ is a symplectic deformation on the trivial ruled surface $\Sigma_g \times S^2$. Our method of detecting nontriviality is based on computations of equivariant Gromov–Witten invariants. One application gives a homotopy-theoretic counterpart to a geometric result obtained by Karshon. Another application concerns the ring structure of $H^*(\mathrm{BSymp}(S^2 \times S^2, \omega_\lambda))$.

1 Introduction

This paper has two parts. In the first, entirely topological, we relate rational Whitehead products to extensions of maps defined on projective spaces, with the intention to apply these results in the second part in a symplectic setting. Namely, we find higher multiplicative Samelson structures in homotopy groups of symplectomorphism groups as

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desuspensions of Whitehead products present in the classifying spaces of the groups under study.

The study of the topology of symplectomorphism groups began with Gromov [11] in 1985 in the case of the projective surface $\mathbb{P}^1 \times \mathbb{P}^1$. Abreu [1], Anjos [4], Abreu–McDuff [3], and more recently, Abreu–Granja–Kitchloo [2] studied in detail the homotopy type of rational ruled surfaces with variable cohomology.

We propose that the appearance of higher Samelson products in the homotopy groups of symplectomorphism groups is in many instances related to the presence of negative self-intersection holomorphic curves detectable by parametric Gromov–Witten invariants.

Whitehead products were first introduced by Porter [19] and rational ones were later studied by Allday [7, 8]. Andrews–Arkovitz [6] studied applications of rational Whitehead products to Sullivan minimal models. Recall that if one considers maps $\eta_i : S^{j_i} \longrightarrow Y$ representing elements in $\pi_* Y$, the kth-order higher Whitehead product $[\eta_{j_1}, \ldots, \eta_{j_k}]$ is a (possibly empty) subset of homotopy classes in $\pi_{r-1} Y$, where $r = j_1 + \ldots + j_k$.

If $a : S^{r-1} \longrightarrow T$ represents the attaching map used to build the product $P = \Pi_{i=1}^k S^{j_i}$ from its $(r-2)$ skeleton T, and i represents the inclusion $S^{j_1} \vee \ldots \vee S^{j_k} \longrightarrow T$, then consider the set $\mathcal{W} := \{\bar{g} : T \longrightarrow Y | \, \bar{g} \circ i = g\}$ of all possible extensions of the wedge map $g = \eta_1 \vee \ldots \vee \eta_k : S^{j_1} \vee \ldots \vee S^{j_k} \to Y$. The kth-order Whitehead product is the set of elements in $\pi_{r-1} Y$ given by the maps $\bar{g} \circ a : S^{r-1} \longrightarrow Y$, for all $\bar{g} \in \mathcal{W}$. Here, $\mathcal{W}$ is nonempty if and only if all the lower Whitehead products contain the element 0. The elements in $[\eta_1, \ldots, \eta_k]$ represent the obstructions to extending all possible maps $\bar{g}$ to the product P. We are interested in all such elements with infinite order that form the rational Whitehead products. They are obtained as Whitehead products in the *the rationalization* $Y_{\emptyset}$ of Y (see Section 2, [6], and references therein).

Definition 1.1. We denote by $W^{(k)}(F) := [F, \cdots, F] \subset \pi_* Y \otimes \mathbb{Q}$ the kth-order Whitehead product of F. If Y is the classifying space of a group G and F is the suspension of an element $\gamma \in \pi_* G$, we will denote by $S^{(k)}(\gamma)$ the corresponding rational Samelson product. □

We find such rational Whitehead in families X_λ, $\lambda \in I$ satisfying the deformation property 2.4, whose compact sets along with homotopies between them extend continuously for a small interval as λ increases in the interval I. If such extensions can be made for the whole length of the interval I, we say that the family has a full deformation property.

To set language, we will introduce the following concepts:

Definition 1.2. We say that a nonzero element $F \in \pi_* X_{\lambda_0}$ is *fragile* if it admits a null-homotopic deformation to the right $0 = F_\lambda \in \pi_* X_\lambda$, for $\lambda > \lambda_0$.

A continuous family $F_\lambda : B \longrightarrow X_\lambda$, $\lambda > \lambda_0$ is *new* with respect to X_{λ_0} if it is not the deformation of a map $F : B \longrightarrow X_{\lambda_0}$.

If the family X_λ has the full deformation property, an element $F \in \pi_* X_{\lambda_0}$ is said to be *robust* if its deformation $F_\lambda \in \pi_* X_\lambda$ is essential (i.e., nonzero).[1] □

Our strategy is to relate Whitehead products to maps $F : \mathbb{P}^n \longrightarrow X_\lambda$ and their multiples given by composites $\mathbb{P}^n \xrightarrow{h} \mathbb{P}^n \xrightarrow{F} X_\lambda$ with h of arbitrary degree. The following proposition gives a strategy to find nontrivial Samelson products[2] in a simple setting where the family X_λ is that of topological groups:

Proposition 1.3. Consider a family of groups G_λ such that G_λ satisfies the deformation property 2.4. Consider a continuous map $\delta : S^1 \longrightarrow G_{\lambda_0}$ and denote by $F_1 : \mathbb{P}^1 \longrightarrow \mathrm{B}G_{\lambda_0}$ its suspension to the classifying space. Assume that the deformations $F_{1,\varepsilon} : \mathbb{P}^1 \longrightarrow \mathrm{B}G_{\lambda_0+\varepsilon}$ admit extensions $F_{2,\varepsilon} : \mathbb{P}^2 \longrightarrow \mathrm{B}G_{\lambda_0+\varepsilon}$ for $\varepsilon > 0$ and that neither they, nor any of their multiples, are homotopic to a map $\mathbb{P}^2 \longrightarrow \mathrm{B}G_{\lambda_0}$. Then one of the following holds:

A) There is a new element of $\pi_3 G_{\lambda_0+\varepsilon} \otimes \mathbb{Q}$.
B) The Samelson product $[\delta, \delta] \in \pi_2 G_{\lambda_0} \otimes \mathbb{Q}$ is nontrivial. □

As an example, the symplectic manifolds $(M, \omega_{1+\varepsilon}) = \big(T^2 \times S^2, (1+\varepsilon)\sigma_{T^2} \oplus \sigma_{S^2}\big)$ $(\varepsilon \geq 0)$, where σ_{S^2} and σ_{T^2} are the areas of the fiber and the base, respectively. We take $G_{1+\varepsilon}$ to be symplectomorphism groups $\mathrm{Symp}\big(T^2 \times S^2, (1+\varepsilon)\sigma_{T^2} \oplus \sigma_{S^2}\big)$. Following McDuff [16], there is a robust family $\delta_{1+\varepsilon} : S^1 \longrightarrow G_{1+\varepsilon}$ $(\varepsilon \geq 0)$ which for $\varepsilon > 0$ is homotopic to a group homomorphism so that its suspension $F_{1,\varepsilon} : \mathbb{P}^1 \longrightarrow \mathrm{B}G_{1+\varepsilon}$ extends over $\mathbb{P}^\infty$.

For these choices of $G_{1+\varepsilon}$ and δ, we use parametric Gromov–Witten invariants in Subsection 4.3 to verify the hypothesis of Proposition 1.3 by showing that the maps $F_{2,\varepsilon}$:

$\mathbb{P}^2 \longrightarrow \mathrm{B}G_{1+\varepsilon}$ ($\varepsilon > 0$) and their multiples are not homotopic to a map $F_2 : \mathbb{P}^2 \longrightarrow \mathrm{B}G_1$ (see Corollary 4.5).

Moreover, because the rational homotopy groups of G_λ are known in this situation, we show in Subsection 4.4 that case A) of Proposition 1.3 does not occur. Hence, we have the following:

Theorem 1.4. There is a fragile element of $\pi_2\mathrm{Symp}(T^2 \times S^2, \sigma_{T^2} \times \sigma_{S^2}) \otimes \mathbb{Q}$ that is represented by the Samelson product $[\delta_1, \delta_1]$. □

Proposition 2.1 in Section 2 introduces a precise setting in which we can single out a nontrivial Whitehead product $W^{(k)}(F)$. Let us present, however, an instance of this proposition which pertains to our main application.

Proposition 1.5. Consider a family of groups G_λ with the deformation property, such that all higher rational Whitehead products in $\mathrm{B}G_{\lambda_0}$ of degree strictly less than $2n$ vanish. Given a continuous homomorphism $\gamma : S^1 \longrightarrow G_{\lambda_0}$, denote by $F_n : \mathbb{P}^n \longrightarrow \mathrm{B}G_{\lambda_0}$ the restriction of the induced map $\mathrm{B}\gamma : \mathrm{B}S^1 \longrightarrow \mathrm{B}G_{\lambda_0}$ to the $2n$ skeleton. Suppose that F_n has a deformation $F_{n,\varepsilon} : \mathbb{P}^n \longrightarrow \mathrm{B}G_{\lambda_0+\varepsilon}$ that extends to $\widetilde{F}_{n,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow \mathrm{B}G_{\lambda_0+\varepsilon}$ when $\varepsilon > 0$, so that condition $(*)$ is satisfied:

$(*)$ No multiple of $\widetilde{F}_{n,\varepsilon}$ is homotopic to a map into $\mathrm{B}G_{\lambda_0}$.

Then one of the following holds

A) There is a new element of $\pi_{2n+2}\mathrm{B}G_{\lambda_0+\varepsilon} \otimes \mathbb{Q}$.

B) There is a nonzero element $W \in \pi_{2n+1}\mathrm{B}G_{\lambda_0} \otimes \mathbb{Q}$ such that the Whitehead product $W^{(n)}(F_1) = \{0, W\}$. □

As above, our example lives in a *symplectic setting*. Let (M, ω_λ) be a smooth family of symplectic structures with a varying cohomology class on a closed manifold M. Let $G = \mathrm{Diff}_0(M)$ be the identity component of the group of diffeomorphisms and $G_\lambda = \mathrm{Symp}(M, \omega_\lambda) \cap \mathrm{Diff}_0(M)$. It follows from Lemma 4.1 in [10] that the family G_λ has the deformation property.

Gromov and Abreu–McDuff have provided information on the additive structure of the homotopy groups $\pi_* G_\lambda \otimes \mathbb{Q}$ in the case that M represents a symplectic ruled surface $(M^g_\lambda, \omega_\lambda) = (\Sigma_g \times S^2, \lambda\sigma_{\Sigma_g} \oplus \sigma_{S^2})$ $(\lambda \geq 1)$ with symplectomorphism groups G^g_λ. Here, σ_{Σ_g} and σ_{S^2} are forms of total area 1. Denote the two homology classes $[\Sigma_g \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ by A and F, respectively. If $k = \lfloor \lambda \rfloor$, then M^g_λ admits k different Hamiltonian circle actions H_i each with two fixed point sets given by holomorphic curves in classes $A \pm iF$, $1 \le i \le k$. Due to the work of Abreu and McDuff [3, 16], appropriate integer multiples of all these actions give the same elements $[\gamma^g]$ in the rational homotopy groups $\pi_1(G^g_\lambda) \otimes \mathbb{Q}$. Using the maps γ^g in Proposition 1.5, we can verify condition $(*)$ for all genus g by using parametric Gromov–Witten (PGW) invariants. In our case, these are equivariant Gromov–Witten (EGW) invariants, and count precisely the H_i-invariant curves. We show that the only *natural* EGW (those counting generically isolated curves with no marked points in certain associated fibrations) are given by

Theorem 1.6. For any genus g and a Hamiltonian circle action with Lie group H_k on M^g_λ $(\lambda > k)$, we have

$$\mathrm{EGW}_{g,0}\big(M^g_\lambda; H_k; s_{A-kF}\big) = \pm 1 \cdot u^{2k+g-1} \in H^*\big(\mathrm{B}S^1, \mathbb{Q}\big). \qquad \square$$

Moreover, because enough information is known about rational homotopy groups in these instances, we can eliminate case A) of Proposition 1.5. One consequence for $g = 1$ was explained in the beginning. In the rational ruled surface case, where the base is a sphere, we obtain the following proposition:

Proposition 1.7. For all $k \ge 1$ and $k < \lambda \le k+1$, and for γ_λ given by the circle action H_k, the Samelson product of order $2k+1$, $S^{(2k+1)}(\gamma_\lambda)$, is equal to $\{0, w_k\} \subset \pi_{4k}(G_\lambda)$, where w_k is a fragile homotopy class that disappears when $\lambda > k+1$. $\square$

Note that the nontriviality of these Whitehead products was established by Abreu–McDuff in [3] using different methods.

We will use this result to give an alternative proof of the following theorem concerning the rational cohomology ring of the classifying spaces $\mathrm{B}G_\lambda$ that is due to Abreu–Granja–Kitchloo [2] (see also Abreu–McDuff [3] for a partial result in this direction).

Theorem 1.8. (Abreu–Granja–Kitchloo) [2] Fix an integer $k \ge 0$. For $k < \lambda \le k+1$, we have the following ring isomorphism $H^*\big(\mathrm{B}G^0_\lambda, \mathbb{Q}\big) = \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle F_k\rangle$, with $\deg \overline{A} = 2$, $\deg \overline{X} = \deg \overline{Y} = 4$, and where the polynomial F_k is given by:

$$F_k = \overline{A}\big(\overline{X} - \overline{Z} + \overline{A}^2\big)(2^2\overline{X} - 2^4\overline{Z} + \overline{A}^2\big) \dots \big(k^2\overline{X} - k^4\overline{Z} + \overline{A}^2\big). \qquad (1)$$

$\square$

For the proof, we use consequences of the existence of a nontrivial rational Whitehead product to the structure of the Sullivan minimal models of the classifying spaces BG^0_λ, along with algebraic computations from [2] that exploit the underlying toric structures of Kähler representatives of $(S^2 \times S^2, \omega_k)$.

In Section 4.7, we discuss the case $g > 1$ and prove the existence of nontrivial higher Whitehead products. As a consequence, we show that the groups G^g_λ do not have the homotopy type of a compact Lie group.

1.1 The structure of the paper

The paper is organized as follows. In Section 2, we introduce and prove the topological result on the existence of nontrivial Whitehead products that enable us to provide examples in the rest of the paper. In the same section, we discuss a relation between rational higher-order Whitehead products and the existence of A_k maps between S^1 and a topological group. In Section 3, we start the discussion of the symplectic setting for our main example when the family X_λ is $\mathrm{BSymp}(M, \omega_\lambda)$. A description of the parametric Gromov–Witten invariants, the main tools we use to determine nontriviality, is provided in Subsection 3.2.

We rephrase Proposition 1.5 in this setting and then focus in Section 4 on our main objects of study, the symplectic ruled surfaces. In 4.1, we provide the necessary background on ruled surfaces. The remaining three Subsections 4.4, 4.5, and 4.7 will study the cases $g = 0$, $g = 1$, and $g > 1$. In particular, for the rational case, we provide in Subsection 4.6 the arguments to obtain an alternative method for the full multiplicative structure of the cohomology rings of the classifying spaces and compare them with [2].

In 4.8, we provide a proof of Theorem 1.6 based on obstruction bundle techniques. We compute there all the natural equivariant Gromov–Witten invariants in the topologically trivial ruled symplectic surfaces.

2 Topological Setting

In this section, we first introduce Proposition 2.1 which describes relations between rational Whitehead products in a space Y and maps from projective spaces $\mathbb{P}^k$ to Y. Afterward, we use this result to study under what conditions families of maps $F_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow X_{\lambda_0+\varepsilon}$, $\varepsilon > 0$ yield nontrivial Whitehead products.

If one considers the rationalization X_0 of X, there exist localization maps $e : X \longrightarrow X_\emptyset$ such that for any $x \in [e_*\eta_1, \ldots, e_*\eta_k]_w \subset \pi_N X_\emptyset$ there are integer numbers M satisfying $M_0 x = e_* z$, with $z \in [M_1\eta_1, \ldots, M_k\eta_k]_w$.

The Whitehead products between elements $e_*\eta$ in the rationalization $X_\emptyset$ are called *rational Whitehead products*. These products are multilinear: if $x \in [e_*\eta_1, \dots, e_*\eta_k]_w$, then $Mx \in [e_*\eta_1, \dots, Me_*\eta_i, \dots, e_*\eta_k]_w$. In light of the above correspondence, and since we will be interested in nontrivial elements of infinite order up to a factor, we will say that the rational Whitehead products considered are of elements in $\pi_* \mathrm{B}G \otimes \mathbb{Q}$. This correspondence can be formalized by considering other definitions of Whitehead products; see for instance Allday [7], who defines rational Whitehead products on the graded differential Lie algebra $\pi_* \mathrm{B}G_\lambda \otimes \mathbb{Q}$.

Proposition 2.1. The following holds:

a) Assume that there is a map $F : \mathbb{P}^1 \longrightarrow Y$ that yields a nontrivial $F \in \pi_2(Y) \otimes \mathbb{Q}$ for which the following rational Whitehead products vanish:

$$\{0\} = W^{(k)}(F) \otimes \mathbb{Q}, k \leq n. \tag{2}$$

Then there is a map $F_n : \mathbb{P}^n \longrightarrow Y$ which extends a map homotopic to a multiple of F.

b) The converse is also true; namely, if there is some $F'_n : \mathbb{P}^n \longrightarrow Y$ that extends a map homotopic to a multiple of F then $0 \in W^{(k)}(F)$, $k \leq n$ and hence $W^{(n+1)}(F)$ is defined. Moreover, if neither F'_n nor one of its multiples can be extended to a map defined on $\mathbb{P}^{n+1}$ then $W^{(n+1)}(F)$ contains a nontrivial element. □

Proof of Proposition 2.1 We will investigate the correspondence between maps defined on $P^{(k)} := (S^2)^k$ and its skeletons and $\mathbb{P}^k$ and its skeletons. We reserve the notation $T^{(k)}$ for the codimension-two (or the fat wedge) skeleton of $P^{(k)}$.

Recall that there is a covering map

$$\mathrm{pr}_{(k)} : P^{(k)} \longrightarrow \mathbb{P}^k = P^{(k)}/\mathcal{S}_k, \tag{3}$$

where $\mathcal{S}_k$ is the kth group of permutations. Then $a_{(k)} : S^{2k-1} \longrightarrow T^{(k)}$ is the universal Whitehead product map, used to attach the top cell of dimension $2k$ on $T^{(k)}$ to obtain $P^{(k)}$. If we look at the Hopf fibration

$$S^1 \longrightarrow S^{2k+1} \xrightarrow{h_{(k)}} \mathbb{P}^k$$

we obtain the Hopf maps $h_{(k)} : S^{2k+1} \longrightarrow \mathbb{P}^k$ used to attach a $(2k+2)$-dimensional cell to $\mathbb{P}^k$ in order to obtain $\mathbb{P}^{k+1}$. The long exact homotopy sequence for this fibration immediately yields that $\pi_{2k+1}\mathbb{P}^k = \mathbb{Z}$ and this group is generated by $h_{(k)}$. Hence for any map $g : S^{2k+1} \longrightarrow \mathbb{P}^k$, we have $[g] = N[h_{(k)}]$, and the number $N \in \mathbb{Z}$ is called the *Hopf invariant* of the map g. Note that if one considers the mapping cone C_g of such maps and denotes by x and z the generators of $H^2(C_g; \mathbb{Z}) = \mathbb{Z}$ and $H^{2k+2}(C_g; \mathbb{Z}) = \mathbb{Z}$, then the Hopf invariant is also given by $x^{k+1} = Nz$. This can be seen by looking at the following cofiber sequence (obtained as a consequence of the fact that $\pi_{2k+1}\mathbb{P}^k = \mathbb{Z}$ and that the mapping cone of the Hopf map $h_{(k)}$ is $\mathbb{P}^{k+1}$):

$$\begin{array}{ccccccc} S^{2k+1} & \xrightarrow{h_{(k)}} & \mathbb{P}^k & \longrightarrow & \mathbb{P}^{k+1} & \longrightarrow & S^{2k+2} \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow{\scriptstyle N} \\ S^{2k+1} & \xrightarrow{g} & \mathbb{P}^k & \longrightarrow & C_g & \longrightarrow & S^{2k+2}. \end{array} \tag{4}$$

It is often easier to compute the Hopf invariant using the cup product in C_g. The following lemma computes the Hopf invariant of the maps $\mathrm{pr}_{(k+1)|T^{(k+1)}} \circ a_{(k+1)}$:

Lemma 2.2. With the above notation, we have $[\mathrm{pr}_{(k+1)|T^{(k+1)}} \circ a_{(k+1)}] = (k+1)![h_{(k)}]$. □

Proof of Lemma 2.2. The Hopf invariant M of the map $\mathrm{pr}_{(k+1)|T^{(k+1)}} \circ a_{(k+1)}$ is obtained from the following cofiber sequence:

$$\begin{array}{ccccccc} S^{2k+1} & \xrightarrow{a_{(k+1)}} & T^{(k+1)} & \xrightarrow{i} & P^{(k+1)} & \longrightarrow & S^{2k+2} \\ \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle \mathrm{pr}_{(k+1)}} & & \downarrow{\scriptstyle \mathrm{pr}_{(k+1)}} & & \downarrow{\scriptstyle M} \\ S^{2k+1} & \xrightarrow{h_{(k)}} & \mathbb{P}^k & \longrightarrow & \mathbb{P}^{k+1} & \longrightarrow & S^{2k+2}. \end{array} \tag{5}$$

If we call $x \in H^2(\mathbb{P}^{k+1}; \mathbb{Z}) = \mathbb{Z}$, $x_i \in H^2(P^{(k+1)}; \mathbb{Z}) = \mathbb{Z}^{k+1}$ the generators of the corresponding cohomologies, then $\mathrm{pr}^*_{(k+1)}(x) = (x_1 + \ldots + x_{k+1})$, and hence $\mathrm{pr}^*_{(k+1)}(x^k) = (k+1)!(x_1 \times \ldots \times x_{k+1})$. Since the degree of the map M must match $(k+1)!$ (as above), the lemma follows. ■

We now proceed to prove part (a) of Proposition 2.1. Fix $n > 1$ and F such that $0 = W^{(k)}(F) \otimes \mathbb{Q}$ for all $k \leq n$. Therefore, the wedge map $F \vee \ldots \vee F$ admits extensions

$$g_{(k)} : T^{(k)} \longrightarrow Y, 1 \leq k \leq n \tag{6}$$

which commute with all possible inclusions $i : T^{(k)} \longrightarrow T^{(k+j)}$.

The following lemma that yields immediately part (a) of Proposition 2.1:

Lemma 2.3.

(1) The maps $g_{(k)}$ in (6) can be chosen *symmetric* in the sense that they commute with the $\mathcal{S}_k$ action. Moreover, they extend to symmetric maps

$$g^{\text{ext}}_{(k)} : P^{(k)} \longrightarrow Y. \tag{7}$$

(2) There exist maps $F_k : \mathbb{P}^k \longrightarrow Y$ that commute with the inclusions $i : \mathbb{P}^k \longrightarrow \mathbb{P}^{k+j}$, whose restriction F_1 satisfies $F_1 = N \cdot F$, a multiple of F and for which we have

$$g^{\text{ext}}_{(k)} = F_k \circ \text{pr}_{(k)} \tag{8}$$

and

$$g_{(k)} = F_{k-1} \circ \text{pr}_{(k)} | T^{(k)}. \tag{9}$$

□

Proof of Lemma 2.3. The following commuting diagram:

$$\begin{array}{ccc} S^{2k+1} & \xrightarrow{N^{k+1}} & S^{2k+1} \\ \downarrow{\scriptstyle h_{(k)}} & & \downarrow{\scriptstyle h_{(k)}} \\ \mathbb{P}^k & \xrightarrow{N} & \mathbb{P}^{k+1} \end{array} \tag{10}$$

implies that $[(N \cdot F)_k \circ h_{(k)}] = N^{k+1}[F_k \circ h_{(k)}]$. Hence, we can kill torsion in $[F_k \circ h_{(k)}]$ by replacing F with its multiple $N \cdot F$. We can then use induction:

(Proof of Lemma 2.3 for $k = 2$) Take $g_{(2)} = F \vee F$, which is clearly symmetric. The obstruction to extending the map $F_{(1)} := F$ from $S^2 = \mathbb{P}^1$ to $\mathbb{P}^2$ is given by the homotopy class $[F \circ h_{(1)}] \in \pi_3 Y$. We have by Lemma 2.2:

$$0 = W^{(2)}(F) = [g_{(2)} \circ a_{(2)}] = [F \circ \text{pr}_{(2)|T^{(2)}} \circ a_{(2)}] = 2[F \circ h_{(1)}]. \tag{11}$$

We replace F by $2F$ and this will kill the two torsion in $[F \circ h_{(1)}]$ because, as above, $[(2F) \circ h_{(1)}] = 4[F \circ h_{(1)}]$. Therefore, after replacing F with its multiple we can extend F_1 to a map $F_2 : \mathbb{P}^2 \longrightarrow Y$. Then, we take the map $g^{\text{ext}}_{(2)} : P^{(2)} \longrightarrow Y$ to be $g^{\text{ext}}_{(2)} = F_2 \circ \text{pr}_{(2)}$, which is clearly symmetric and extends $g_{(2)}$.

(Proof that Lemma 2.3 for k implies Lemma 2.3 for $k+1$) We have that $T^{(k+1)} = \bigcup_{j=0}^{j=k} P^{(k)}_j$ where $P^{(k)}_j$ is an identification of the product $P^{(k)}$ with the space of $(k+1)$-tuples that have the coordinate in position j at the base point x_j. The spaces $P^{(k)}_j$ intersect in spaces homeomorphic to $P^{(k-1)}$. By the induction step, we already have $k+1$ copies $g^{j,\text{ext}}_{(k)}$ of the symmetric map $g^{\text{ext}}_{(k)}$ with domains $P^{(k)}_j$, as well as a map F_k with $g^{\text{ext}}_{(k)} = F_k \circ \text{pr}_{(k)}$. Since all the maps $g^{j,\text{ext}}_{(k)}$ agree on intersections, we define $g_{(k+1)} : T^{(k+1)} \longrightarrow Y$ by

$$g_{(k+1)} := \bigcup_{j=0}^{k} g^{j,\text{ext}}_{(k)}.$$

Clearly, $g_{(k+1)} = F_{(k)} \circ \text{pr}_{(k+1)|T^{(k+1)}}$. Moreover, by hypothesis, the obstruction to extend this map to the product is $0 = [g_{(k+1)} \circ a_{(k+1)}]$ and using Lemma 2.2 we get $(k+1)![F_k \circ h_{(k)}] = [g_{(k+1)} \circ a_{(k+1)}]$. By replacing F with $N \cdot F$, we can kill the undesired torsion in $[F_k \circ h_{(k)}]$. Hence, we may conclude that $[F_k \circ h_{(k)}] = 0$ and therefore the map F_k can be extended to $F_{k+1} : \mathbb{P}^{k+1} \longrightarrow Y$. As before, we define $g^{\text{ext}}_{(k+1)} = F_{k+1} \circ \text{pr}_{(k+1)}$, which is a symmetric extension of $g_{(k+1)}$.

This concludes the proof of Lemma 2.3. ∎

To prove (b) of Proposition 2.1, let us consider the maps $F'_n \circ \text{pr}_{(n+1)|T^{(n+1)}} : T^{(n+1)} \longrightarrow Y$. They provide an extension of $F'_1 \vee \ldots \vee F'_1$ to $T^{(n+1)}$ and hence $W^{(n+1)}(F'_1)$ is defined, thus $W^{(n+1)}(F) \otimes \mathbb{Q}$ is also defined. Let us assume that $F'_n : \mathbb{P}^n \longrightarrow Y$ cannot be extended over $\mathbb{P}^{n+1}$. This implies that $[F' \circ h_{(n)}] \neq 0$. We know that the obstruction to extend the map $g_{(n)}$ satisfies $[g_{(n)} \circ a_{(n)}] = (n+1)![F'_n \circ h_{(n)}]$. Again, if we work rationally we can insure by considering a multiple of F'_n that the homotopy class $[F_n \circ h_{(n)}]$ is of infinite order and hence $[g_{(n+1)} \circ a_{(n+1)}] \neq 0$ which gives a nontrivial $W^{(n+1)}(F'_1)$. ∎

2.1 Deformations of Whitehead products

Now consider a topological family $X_{\lambda_0+\varepsilon}$ that admits the following deformation property:

Definition 2.4. Let $I \subset \mathbb{R}$ be an open interval and X_λ, $\lambda \in I$, be a family of closed subspaces of a topological space X. For $J \subset I$, let $X_J = \cup_{\lambda \in J}\{\lambda\} \times X_\lambda \subset J \times X$. We say that the family $\{X_\lambda\}$ has the *deformation property* if for each $\lambda \in I$ and every compact set

$K \subset X_\lambda$ there is $\varepsilon_K > 0$ and a continuous map $h : [0, \varepsilon_K] \times K \longrightarrow X_{[\lambda,\lambda+\varepsilon_K]}$ such that the following diagram commutes

$$\begin{array}{ccc} h : [0, \varepsilon_K] \times K & \longrightarrow & X_{[0,\infty)} \\ \Big\downarrow \mathrm{pr}_1 & & \Big\downarrow \mathrm{pr}_2 \\ [0, \varepsilon_K] & \xrightarrow{\ \mathrm{incl}\ } & [0, \infty). \end{array} \tag{12}$$

Moreover, the germ of the deformation h is unique up to homotopy in the sense that for any two maps h and h' satisfying this diagram and which coincide on $0 \times K$, there exists, for small enough $\varepsilon' > 0$, a homotopy $H : [0, 1] \times [0, \varepsilon'] \times K \to X_{[0,\infty)}$ between h and h' which also satisfies $H \circ \mathrm{pr}_2 = \mathrm{pr}_1 \circ \mathrm{incl}$.

We say that the family X_λ, $\lambda \in I$ satisfies the *full deformation property* if both h and H exist for all $\varepsilon > 0$. □

Proposition 2.5. For $F : \mathbb{P}^1 \longrightarrow X_{\lambda_0}$, if $W^{(k)}(F)$ is defined, then $W^{(k)}(F_\varepsilon)$ is also defined for small positive ε and for F_ε a deformation of F. □

The proof is immediate. Let $Y = X_{\lambda_0}$ in Proposition 2.1. Any extension $F_p : \mathbb{P}^n \longrightarrow X_{\lambda_0}$ provided by part (a) has a deformation $F_{n,\varepsilon} : \mathbb{P}^n \longrightarrow X_{\lambda_0+\varepsilon}$ extending a map $F_\varepsilon :\ P^1 \longrightarrow X_{\lambda_0+\varepsilon}$ for small ε.

Corollary 2.6. Assume that we have a map $F_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow X_{\lambda_0+\varepsilon}$, $\varepsilon > 0$ satisfying

I) $F_{n+1,\varepsilon}$ and any of its multiples are **new maps** with respect to X_{λ_0} in the family $X_{\lambda_0+\varepsilon}$, $\varepsilon > 0$.

II) The restriction $F_{n,\varepsilon} : \mathbb{P}^n \longrightarrow X_{\lambda_0+\varepsilon}$ belongs to a continuous family $F_{n,\varepsilon}$, $\varepsilon \geq 0$.

Then one of the following holds

A) There is a **new element** with respect to X_{λ_0} in $\pi_{2n+2}(X_{\lambda_0+\varepsilon}) \otimes \mathbb{Q}$, $\varepsilon > 0$.

B) There is a nonzero element $w \in \pi_{2n+1}(X_{\lambda_0}) \otimes \mathbb{Q}$ in the Whitehead product $W^{(n+1)}(F_1)$. □

In general, we can replace condition II by a weaker condition and obtain:

Corollary 2.7. Assume that we have a map $F_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow X_{\lambda_0+\varepsilon}$, $\varepsilon > 0$ satisfying

I) $F_{n+1,\varepsilon}$ and any of its multiples are **new maps** with respect to X_{λ_0} in the family $X_{\lambda_0+\varepsilon}$, $\varepsilon > 0$.

III) There is a k with $1 < k \leq n$ such that the restriction $F_{k,\varepsilon} : \mathbb{P}^k \longrightarrow X_{\lambda_0+\varepsilon}$ belongs to a continuous family $F_{k,\varepsilon}$, $\varepsilon \geq 0$.

Then one of the following holds

C) There is a **new element** with respect to X_{λ_0} in $\pi_{2r+2}(X_{\lambda_0+\varepsilon}) \otimes \mathbb{Q}$, for some r with $k \leq r \leq n$.

D) There is a nonzero element $w \in \pi_{2r+1}(X_{\lambda_0}) \otimes \mathbb{Q}$ in the Whitehead product $W^{(r+1)}(F_1)$, for some r with $k \leq r \leq n$. □

Proof of Corollary 2.6. Consider the map $F_{n,0} : \mathbb{P}^n \longrightarrow X_{\lambda_0}$ provided by assumption II. If neither $F_{n,0}$ nor its multiples can be extended to a map defined on $\mathbb{P}^{n+1}$ then, using part b) of Proposition 2.1, situation B) will hold.

Otherwise, consider an extension $F_{n+1,0} : \mathbb{P}^{n+1} \longrightarrow X_{\lambda_0}$. This map will deform to a family $F'_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow X_{\lambda_0+\varepsilon}$ for sufficiently small ε. We will now have two maps, $F'_{n+1,\varepsilon}$ and the original map $F_{n+1,\varepsilon}$, whose restrictions to the codimension-two skeletons $\mathbb{P}^n$ are homotopic. But $F'_{n+1,\varepsilon}$ and $F_{n+1,\varepsilon}$ cannot be homotopic to each other because $F_{n+1,\varepsilon}$ represents a new family. Thus, there must be a new element η_ε in $\pi_{2n+2}(X_{\lambda_0+\varepsilon})$ obtained as follows: after a homotopy we can assume that $F'_{n,\varepsilon}$ and $F_{n,\varepsilon}$ are the same, hence their extensions $F'_{n+1,\varepsilon}$ and $F_{n+1,\varepsilon}$ are built using two maps $\phi_1 : e^{2n+2} \longrightarrow X_{\lambda_0+\varepsilon}$ and $\phi_2 : e^{2n+2} \longrightarrow X_{\lambda_0+\varepsilon}$ that agree when restricted to the boundary of the top cells e^{2n+2}. By gluing these two maps along their boundaries, we obtain a new map $\eta_\varepsilon : S^{2n+2} \longrightarrow X_{\lambda_0+\varepsilon}$, which is not null-homotopic since $F'_{n+1,\varepsilon}$ and $F_{n+1,\varepsilon}$ are not homotopic. By taking sufficient high multiples of the maps F and F', we can ensure that η_ε has no vanishing multiples.

Corollary 2.7 follows immediately as above. ■

2.2 A criterion for the existence of A_k maps between topological groups

Let us offer the following interpretation of Proposition 2.1 for the case when Y represents the classifying space of a topological group G.

Consider a continuous map $f : S^1 \longrightarrow G$ and its suspension $F : \mathbb{P}^1 \longrightarrow BG$. The obstructions for extending F to a map $F_\infty : \mathbb{P}^\infty = BS^1 \longrightarrow BG$ are the same as for extending f as an H-homomorphism (otherwise known as A_∞ map) (see Stasheff [22]). Recall that f is an H-homomorphism if there are two topological groups S, K weakly homotopy equivalent to S^1 and G, respectively, and a group homomorphism $\phi : S \longrightarrow K$ whose homotopy class corresponds to the class of f via the homotopy equivalences involved.

In general, the obstructions to extend F to a map $F_k : \mathbb{P}^k \longrightarrow \mathrm{B}G$ is the obstruction to realize f as an A_k map. So one can interpret parts of Proposition 2.1 as follows:

Corollary 2.8. Consider a continuous map $f : S^1 \longrightarrow G$ and its suspension $F : \mathbb{P}^1 \longrightarrow \mathrm{B}G$. Then a multiple of F can be realized as an A_k map if and only if the rational Whitehead product $W^{k+1}(F)$ is defined. □

This interpretation merely says that if zero is contained in the rational Whitehead products some extensions exist, hence one can find extensions of F; but Proposition 2.1 gives an interpretation of all the homotopy types of such extensions.

Remark 2.9. Observe that the proofs of the results used in Subsection 2.1 only use the deformation property 2.4 of the family X_λ when the compact subset K is a simply connected closure-finite weak topology CW-complex. If we have a family of groups G_λ that satisfy the deformation property 2.4, then the family of classifying spaces $X_\lambda = BG_\lambda$ satisfies a weaker version of definition property 2.4 for K is a simply connected CW complex.

3 Symplectic Setting

Consider (M, ω_λ) a continuous family of symplectic forms with variable cohomology class on a compact manifold M. In the rest of the paper, we will study the symplectomorphism groups $G_\lambda = \mathrm{Symp}(M, \omega_\lambda) \cap \mathrm{Diff}_0(M)$, with $\mathrm{Diff}_0(M)$ the identity component of the diffeomorphism groups. There is no direct inclusion of elements from G_λ in $G_{\lambda+\varepsilon}$. We have nevertheless the following:

Proposition 3.1.

1. (Buşe [10]) The family G_λ, $\lambda \geq 0$, with $G_\lambda \subset \mathcal{G} = \mathrm{Diff}_0(M)$ satisfies the deformation property.
2. The family of classifying spaces $\mathrm{B}G_\lambda$, $\lambda \geq 0$, with $\mathrm{B}G_\lambda \subset \mathrm{B}\mathcal{G} := \mathrm{BDiff}_0(M)$ satisfies the deformation property when K is a simply connected CW complex.

3. (McDuff [16]) Consider the family $M^g_\lambda := (\Sigma_g \times S^2, \lambda\sigma_{\Sigma_g} \oplus \sigma_{S^2})$, where the topologically trivial fibrations with two forms σ_{S^2} and σ_{Σ_g} have total area 1, and consider G^g_λ, their corresponding symplectomorphism groups. Then both families G^g_λ and $\mathrm{B}G^g_\lambda$, $\lambda \geq 1$ have the full deformation property. □

Note that case (2) follows from the second part of Remark 2.9. As we will see later, item (3) of Proposition 3.1 follows from a stronger result of D. McDuff [16]; namely, she provides continuous maps $h_{\lambda,\lambda+\varepsilon} : G^g_\lambda \longrightarrow G^g_{\lambda+\varepsilon}$ that are in fact A_∞ maps and hence induce continuous maps $Bh_{\lambda,\lambda+\varepsilon} : BG^g_\lambda \longrightarrow BG^g_{\lambda+\varepsilon}$. An explicit proof of their property of being A_∞ maps is provided in Abreu–Granja–Kitchloo [2].

We will be interested to apply Proposition 2.1 and its corollaries for such symplectic families $\mathrm{B}G_\lambda$ in general. Nevertheless, the existence of essential robust elements with sufficiently many trivial Whitehead products is difficult to prove in general and hence we will ultimately reduce to the case of ruled surfaces to provide examples.

Let us observe that a map $f : B \longrightarrow \mathrm{BSymp}(M, \omega_\lambda)$ is the same as a *symplectic fibration* with base B and fiber M, whose structural group is inside the symplectomorphism group. Thus, studying extensions of such maps will in fact mean studying symplectic fibrations. Let us introduce some preliminaries for convenience.

3.1 Symplectic fibrations

Consider a triple (M, ω_0, J) where J is an almost complex structure that tames ω_0 and has a canonical class $c_1(M)$.

Definition 3.2. A locally trivial fibration $\pi : Q \longrightarrow B$ is a **symplectic fibration** if the fiber is a compact symplectic manifold (M, ω_0) and there exists a two-form Λ_0 on Q which is vertically closed (i.e., $i(v_1, v_2)d\Lambda_0 = 0$ for all vertical vectors v_i) and whose restriction to each fiber is the symplectic form of the fiber. □

As shown in [12], such forms correspond to symplectic connections on the fibration. Consider (U_α) an atlas covering the base B and a trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow M \times U_\alpha$, that yields a collection of transition maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Diff}(M)$. An equivalent definition of the symplectic fibration is that $\phi_{\alpha\beta} \subset \mathrm{Symp}(M, \omega_0)$. Indeed, given such a trivialization, the form Λ_0 is obtained via a partition of unity from canonical forms on $\pi^{-1}(U_\alpha)$ such that it restricts on each fiber M_b to $\omega_b = \phi_\alpha|^*_{\pi^{-1}(b)}\omega_0$.

Given a symplectic fibration, we consider the *associated cohomological and homological bundles* $H^*(M,\mathbb{R}) \longrightarrow Q^* \longrightarrow B$ and $H_*(M,\mathbb{Z}) \longrightarrow Q_* \longrightarrow B$. These are obtained by considering the same atlas for the base and automorphisms naturally induced by the maps ϕ_α on homology, respectively cohomology.

In a similar manner, one constructs an associated bundle $\mathcal{J}(B, M)$ whose fiber over each b is the space of almost complex structures J on M that are tamed by ω_b. As explained in Le–Ono [14], since the fibers are contractible, one can always pick a section $b \longrightarrow J_b$ in this bundle.

The above alternative descriptions of a symplectic fibration imply that there exist constant sections $s^{[\omega_0]} : B \longrightarrow Q^*$ with the value $[\omega_0] \in H^*(M,\mathbb{R})$ and $s^{[c_1]} : B \longrightarrow Q^*$ that take the integer value $c_1(M, \omega_0) \in H^2(M,\mathbb{R})$.

We say that a symplectic fibration is a **Hamiltonian fibration** if the structure group further reduces to $\mathrm{Ham}(\mathrm{M}, \omega_0) \subset \mathrm{Symp}(\mathrm{M}, \omega_0)$.

By a result of Guillemin et al. [12], a symplectic fibration with a simply connected base B is Hamiltonian if and only if there exists a *closed extension* $\Lambda_0 \in \Omega^2(Q)$. Moreover, a result of Thurston [17, page 197] guarantees that if the base B carries a symplectic form σ_B, then for t sufficiently large the form Λ_0 can be chosen to be symplectic and to represent the class $[\omega_0] + t[\pi^*\sigma_B]$.

If $\pi_1(B)$ acts trivially on the associated fibration $H_*(M,\mathbb{Z}) \longrightarrow Q_* \longrightarrow B$ (e.g., if B is simply connected), then for each $D \in H_2(M,\mathbb{Z})$ there also exists a constant section $s_D : B \longrightarrow Q_*$ that takes the value D.

Let us consider a symplectic deformation $(M, \omega_\lambda)_{\lambda \geq 0}$ of the symplectic structure (M, ω_0).

Definition 3.3. We say that a continuous one parameter family of vertically closed two forms $(\Lambda_\lambda)_{\lambda \geq 0}$ on Q that satisfy the conditions in Definition 3.2 for symplectic fibers (M, ω_λ), represents a *fiberwise symplectic deformation based on* the family $(M, \omega_\lambda)_{\lambda \geq 0}$. □

These fibrations carry *vertical almost complex structures* $\widetilde{J}_\lambda$. These are automorphisms of the vertical tangent bundle such that $\widetilde{J}_\lambda{}^2 = -Id$. We say $\widetilde{J}_\lambda$ is *compatible* with Λ_λ if $\Lambda_\lambda(\cdot, \widetilde{J}_\lambda \cdot)$ provides a metric in each fiber M_b.

We will refer to such pairs $(\Lambda_\lambda, \widetilde{J}_\lambda)$ as compatible with the symplectic fibration Q with fiber (M, ω_λ).

We are interested in applying Corollaries 2.6 and 2.7 for the cases $X_\lambda = \mathrm{BSymp}(M, \omega_\lambda)$. Let us point out that a deformation $F_{n,\varepsilon} : \mathbb{P}^n \longrightarrow \mathrm{BSymp}(M, \omega_{\lambda_0+\varepsilon})$, $\varepsilon \geq 0$, corresponds to a fiberwise symplectic deformation $(\Lambda_\lambda)_{\lambda\geq 0}$ on $Q \longrightarrow \mathbb{P}^n$ based on the family $(M, \omega_\lambda)_{\lambda\geq 0}$. Thus, the essential tool to show that a given family $F_{n+1,\varepsilon}$, $\varepsilon > 0$ is a new family as in part I of Corollary 2.6 must in fact be a tool that shows obstructions to deform it to $\varepsilon = 0$. These tools will be parametric Gromov–Witten invariants, which are introduced in the next section. The informed reader may skip and go to Subsection 3.3 to see them applied to our setting.

3.2 Definition and properties of parametric Gromov–Witten invariants

We will first make a summary of their defining properties. We will use results from Li–Tian [15] as well as results from Le–Ono [14].

Assume that the symplectic fibration $\pi : Q \longrightarrow B$ with fiber (M, ω) admits a closed extension Λ of ω. As explained in Subsection 3.1, we may consider a section $\widetilde{J} : B \longrightarrow \mathcal{J}(B, M)$ that provides an almost complex structure on each fiber M_b compatible with the symplectic form ω_b. It will suffice for our purposes to consider B a simply connected compact space.

For $2g + m \geq 3$, let $\mathcal{M}_{g,m}$ be the moduli space of genus g Riemann surfaces $(\Sigma_g, x_1 \cdots x_m)$ (up to biholomorphisms of marked surfaces) with m marked distinct points. As usual, the $(3g - 3 + m)$-dimensional Kähler orbifold $\overline{\mathcal{M}}_{g,m}$ is the Deligne–Mumford compactification of $\mathcal{M}_{g,m}$. This consists of all genus g stable curves (up to biholomorphisms of marked surfaces taking nodes to nodes) with at most rational double points different from the m marked points.

Fix a homology class $D \in H_2(M, \mathbb{Z})$. Since we assumed that B simply connected, we get a constant section $s_D : B \longrightarrow H_2(Q, \mathbb{Z})$ in the corresponding homological bundle.

Definition 3.4. A *vertical stable C^l-map* $(b, f, x_1, \cdots x_m)$ with m marked points is a map $f : (\Sigma_g, x_1 \cdots x_m) \longrightarrow Q$ whose image is contained in some fiber Q_b and satisfies the following conditions:

(1) Σ is a connected (possibly singular) curve with normal crossings and $x_1 \cdots x_m$ are smooth distinct points on Σ.
(2) f is continuous and each restriction to an irreducible component $f_{|\Sigma_i}$ lifts to a C^l-smooth map from the normalization $\widetilde{\Sigma_i}$ into Q.
(3) Any irreducible component Σ_i of genus 0 on which f is constant must contain at least three *special points* (that are either marked points or singular points of Σ). □

Note that the condition $2g+m \geq 3$ is not imposed in Definition 3.4. If $2g+m \geq 3$, the domain curve $(\Sigma_g, x_1 \cdots x_m)$ is stable if its group of biholomorphic self-maps $Aut(\Sigma, x_1, \cdots x_m)$ is finite. If one looks at $Aut(b, f, \Sigma, x_1, \cdots x_m)$ satisfying $f \circ \sigma = f$, then stability of the map $(b, f, x_1, \cdots x_m)$ implies only finiteness of the latter.

Consider $f : (\Sigma, x_1 \cdots x_m) \longrightarrow Q$ and $f' : (\Sigma', x_1' \cdots x_m') \longrightarrow Q$. We say that two stable maps $(b, f, x_1, \cdots x_m)$ are equivalent if $b = b'$, both $\mathrm{im}(f)$ and $\mathrm{im}(f')$ are contained in the same fiber Q_b, and there is a biholomorphism of the domains $\phi : \Sigma \longrightarrow \Sigma'$ that takes marked points to marked points, nodal points to nodal points (and hence irreducible components to irreducible components), and such that $f \circ \phi = f'$. Let $\overline{\mathcal{F}}^l_{g,m}(Q, s_D)$ be the moduli space of equivalence classes $[b, f, x_1, \cdots, x_m]$ as above such that f is C^l smooth and $[\mathrm{im}(f)] = s_D(b) \in H_2(Q_b, \mathbb{Z})$.

A vertical map f with $\mathrm{im}(f) \subset Q_b$ is J_b holomorphic if there is an arbitrary complex structure $j \in \mathrm{Teich}(\Sigma)$ on Σ, such that $\bar{\partial}_{J_b}(f) = \frac{1}{2}(df + J_b \circ df \circ j) = 0$. We denote by $\overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D)$ the subset of $\overline{\mathcal{F}}^l_{g,m}(Q, s_D)$ consisting of J_b-holomorphic stable maps.

If Σ is smooth, we denote by $\Omega^{(0,1)}(f^* TQ_b^{\mathrm{vert}})$ the set of all continuous sections ξ in $\mathrm{Hom}((T\Sigma), f^* TQ_b^{\mathrm{vert}})$ that anti-commute with j and J_b. Let $\mathrm{Reg}(\Sigma) \subset \Sigma$ be the set of all nonsingular points of Σ. In general, $\Omega^{(0,1)}(f^* TQ_b^{\mathrm{vert}})$ consists of $f^* TQ_b^{\mathrm{vert}}$-valued $(0, 1)$ forms ξ over $\mathrm{Reg}(\Sigma)$ that have the property that the restrictions $\xi_{|\Sigma_1}$, $\xi_{|\Sigma_2}$ of two local components near a singular point q can be extended continuously across q.

We can construct a generalized bundle E over $\overline{\mathcal{F}}^l_{g,m}(Q, s_D)$ with fiber $\Omega^{(0,1)}(f^* TQ_b^{\mathrm{vert}})$ and consider a section in E given by $\Phi = \frac{1}{2}(df + J_b \circ df \circ j)$. Then, $\Phi^{-1}(0)$ is exactly $\overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D)$.[3]

Proposition 3.5. For $l \geq 2$ and the section $\phi : \mathcal{F}^l_{g,m}(Q, s_D) \longrightarrow E$ as above, $\phi^{-1}(0) = \overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D)$ is compact and ϕ gives rise to a generalized Fredholm orbifold bundle with a natural orientation and index $d = 2(\dim_{\mathbb{C}} M - 3)(1-g) + 2c_1(D) + 2m + \dim B$. □

Following the same line of argument as in [15], the above result allows one to construct a *virtual moduli class* $[\overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D)]^{\mathrm{virt}} \in H_d(\overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D), \mathbb{Q})$. Let us consider now the usual evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D) \longrightarrow Q^m$$

given by $\mathrm{ev}([b, f, x_1, \cdots x_m]) = (f(x_1), \cdots, f(x_m))$, as well as the forgetful map $\mathrm{forget} : \overline{\mathcal{M}}_{g,m}(Q, \tilde{J}, s_D) \longrightarrow \overline{\mathcal{M}}_{g,m}$, whose value is the stabilized domain (collapsing unstable components) of f. In what follows, we define $\overline{\mathcal{M}}_{g,m}$ to be a point whenever $2g+m<3$.

Definition 3.6. The parametric Gromov–Witten invariants are maps

$$\mathrm{PGW}_{g,m}(Q,s_D):[H^*(Q;\mathbb{Q})]^m\times H^*(\overline{\mathcal{M}}_{g,m},\mathbb{Q})\longrightarrow\mathbb{Q} \tag{13}$$

which, for $\alpha\in[H^*(Q;\mathbb{R})]^m$ and $\beta\in H^*\overline{\mathcal{M}}_{g,m}$ are given as:

$$\mathrm{PGW}_{g,m}(Q,s_D)(\alpha,\beta)=[\overline{\mathcal{M}}_{g,m}(Q,\tilde{J},s_D)]^{\mathrm{virt}}\cap(\mathrm{forget}^*\beta\cup\mathrm{ev}^*\alpha).$$

These invariants are zero unless

$$2(\dim_{\mathbb{C}}M-3)(1-g)+2c_1(D)+2m+\dim B=\dim\alpha+\dim\beta. \tag{14}$$

□

Let us focus on the case when $\beta=1$ and α is the Poincaré dual of a product of m cycles a_i that can be represented in a fiber Q_b for some arbitrary b. Then the invariants count all maps $[b,f,j,x_1,\dots,x_m]$ whose homology class is $[\mathrm{im}(f)]=s_D\in H_2(Q_b,\mathbb{Z})$ and such that $f(x_i)$ lies in a_i.

We define *the symplectic vertical taming cone* $\mathcal{T}(\widetilde{J})$ of a section $\widetilde{J}$ to be the space of closed two forms Λ on Q that are compatible with the symplectic fibration $\pi:Q\longrightarrow B$ with fiber (M,ω) and which satisfy the taming relation $\Lambda(v,\widetilde{J}v)>0$ for any vector v tangent to a fiber Q_b.

As in Li–Tian [15] and Le–Ono [14], the following properties of parametric Gromov–Witten invariants hold:

Proposition 3.7. (Properties of parametric Gromov–Witten invariants). Consider a symplectic fibration $\pi:Q\longrightarrow B$ with fiber (M,ω_0), with a closed extension Λ_0 of ω_0 and an integral homology class $D\in H_2(M,Z)$.

(i) The parametric Gromov–Witten invariants $\mathrm{PGW}_{g,m}(Q,s_D)$ are well defined and independent of the choice of the section of tamed vertical almost complex structure $\widetilde{J}$ with $\Lambda_0\in\mathcal{T}(\widetilde{J})$.

(ii) The parametric Gromov–Witten invariants $\mathrm{PGW}_{g,m}(Q, s_D)$ are independent of the choice of the *taming* closed extension Λ_0 and hence are **fiberwise symplectic deformation invariants** as long as the deformation is within some symplectic taming cone $\mathcal{T}(\widetilde{J})$.

(iii) (Le–Ono [14]) **Symplectic sum formula**: let $Q = Q_1 \# Q_2$ be a fiber connected sum of two fibrations. Then

$$\mathrm{PGW}_{g,0}(Q, s_D) = \mathrm{PGW}_{g,0}(Q_1, s_D) + \mathrm{PGW}_{g,0}(Q_2, s_D). \tag{15}$$

□

(iv) (Le–Ono [14]) If $f : B' \longrightarrow B$ is a N covering map, then $\mathrm{PGW}_{g,m}(Q, s_D) = N \cdot \mathrm{PGW}_{g,m}(f^* Q, s'_D)$.

3.2.1 *Equivariant Gromov–Witten invariants*

Equivariant Gromov–Witten invariants can be defined for any Hamiltonian action of a compact Lie group H on a symplectic manifold (M, ω). We will restrict ourselves to the case of Hamiltonian circle actions $H = S^1$. To define them, we will follow here the approach of Givental, Givental–Kim, and Ruan; namely, equivariant Gromov–Witten (GW) invariants will be viewed as limits of parametric GW-invariants.

Consider the universal symplectic fibration $M_{S^1} = M \times_{S^1} ES^1$ with fiber (M, ω). M_{S^1} consists of an infinite tower of Hamiltonian fibrations $\pi_{(k)} : M^{(k)}_{S^1} = M \times_{S^1} S^{2k+1} \longrightarrow \mathbb{C}P^k$. Note that M comes equipped with an S^1-invariant symplectic form ω. By taking its pullback to the product $M \times ES^1$ and its descendants to the quotients $M^{(k)}_{S^1}$, we obtain closed two-form extensions $\Lambda^{(k)}$. Similarly,using an S^1-invariant compatible almost complex structure on M, we obtain a natural vertical almost complex structure $J^{(k)}$ compatible with the fibration that makes the map $\pi_{(k)}$ almost holomorphic, so that the triple $(M^{(k)}_{S^1}, J^{(k)}, \Lambda^{(k)})$ is compatible with the fibration $\pi_{(k)} : M^{(k)}_{S^1} \longrightarrow \mathbb{C}P^k$ with fibers (M, ω).

We say that M_{S^1} admits the vertical almost complex structure $\widetilde{J}$ if $\widetilde{J}$ restricts to the usual vertical almost complex structure on each $M^{(k)}_{S^1}$. Similarly, we say that Λ is a closed two form on M_{S^1} if it restricts to a closed two form on each $M^{(k)}_{S^1}$.

Then, the equivariant Gromov–Witten invariants are maps

$$\mathrm{EGW}_{g,m}(M, s_D) : [H^*(M_{S^1}, \mathbb{Q})]^m \times H^*(\overline{\mathcal{M}}_{g,m}, \mathbb{Q}) \longrightarrow H^*(\mathrm{B}S^1, \mathbb{Q}) \tag{16}$$

which are defined as follows:

Take $\alpha \in [H^*(M_{S^1}, \mathbb{Q})]^m$, $\beta \in \overline{\mathcal{M}}_{g,m}$. Note that since

$$H^*(\mathrm{B}S^1, \mathbb{Q}) = H^*(\mathbb{C}P^\infty, \mathbb{Q}) = Q[[u]],$$

u of degree 2, any class in $\gamma \in H^*(\mathrm{B}S^1, \mathbb{Q})$ can be written as $\gamma = \Sigma_{k=1}^{k=\infty} \gamma_k u^k$. Consider the inclusion maps $i_{M^{(k)}_{S^1}} : (M^{(k)}_{S^1})^m \longrightarrow (M_{S^1})^m$. Then we have

$$\mathrm{EGW}_{g,m}(M, s_D)(\beta, \alpha) = \bigoplus_{k=1}^{k=\infty} \mathrm{EGW}^{(k)}_{g,m}(M^{(k)}_{S^1}, s_D)(i^*_{M^{(k)}_{S^1}}\alpha, \beta) u^k \tag{17}$$

where $\mathrm{EGW}^{(k)}_{g,m}(Q^{(k)}, s_D)(\alpha^{(k)}, \beta)$ represent parametric Gromov–Witten invariants of the fibration $M^{(k)}_{S^1}$ that are zero unless

$$2(\dim_{\mathbb{C}} M - 3)(1 - g) + 2c_1(D) + 2m + 2k = \dim i^*_{M^{(k)}_{S^1}}\alpha + \dim \beta. \tag{18}$$

Remark 3.8. Our formalism here follows closely the way EGW are introduced by Ruan [20] for a general choice of the acting group H (see Ruan's Definition 5.1 and Theorem 5.2). To obtain a complete identification and hence the properties mentioned below in Proposition 3.9, one has to see that for our case $H = S^1$ the homology $H_*(\mathrm{B}S^1, \mathbb{Z})$ is generated by the classes $[\mathbb{C}P^k]$ for all k and hence via Ruan's [20] Proposition 4.4 we obtain the desired equivalence between our and his definition. □

The following proposition gives properties of equivariant Gromov–Witten invariants:

Proposition 3.9.

1. For any pair $(\Lambda, \widetilde{J})$ compatible with the fibration M_{S^1} with S^1-invariant fiber (M, ω), the invariants $\mathrm{EGW}_{g,m}(M, s_D)$ are well defined and independent of the choice of the taming vertical almost complex structure $\widetilde{J}$.
2. The invariants $\mathrm{EGW}_{g,m}(M, s_D)$ do not change under a symplectic deformation ω_λ that induces a fiberwise symplectic deformation $(M_{S^1}, \Lambda_\lambda)$. □

3.3 Relation between PGW and Whitehead products

Proposition 3.10. Consider a symplectic deformation $(M, \omega_\lambda)_{\lambda \geq 0}$ and a homology class $D \in H_2(M, \mathbb{Z})$ with $[\omega_0](D) = 0$. Assume that there exists a smooth symplectic fibration $\pi : Q \longrightarrow B$ endowed with a continuous family of closed two-form extensions $(\Lambda_\lambda)_{\lambda \geq 0}$ of the symplectic fibers (M, ω_λ). Then, any well defined $\mathrm{PGW}_{g,m}(Q_\lambda, s_D)$ must be trivial. □

An effective tool to find families that satisfy condition I in Corollaries 2.6 and 2.7 is given by:

Corollary 3.11. Consider a symplectic deformation $(M, \omega_\lambda)_{\lambda \geq 0}$ and a homology class $D \in H_2(M, \mathbb{Z})$ with $[\omega_0](D) = 0$. If the fiberwise symplectic deformation corresponding to a continuous family $F_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow \mathrm{BSymp}(M, \omega_{\lambda_0+\varepsilon})$, $\varepsilon > 0$ exhibits a nontrivial $\mathrm{PGW}_{g,m}(Q_{\lambda>0}, s_D)$, then both $F_{n+1,\varepsilon}$ and its multiples must be **new families**. □

Nevertheless, to find families that satisfy *both* conditions I and II is more difficult; that is because one needs to know a great deal more about the additive structure of the symplectomorphism groups to be able to satisfy II. Hence, we restrict our attention to the ruled surfaces in the next section.

4 Ruled Surfaces M_λ^g

A topologically trivial ruled surface M_λ^g is the total space of the topologically trivial symplectic fibration $(\Sigma_g \times S^2, \lambda\sigma_{\Sigma_g} \oplus \sigma_{S^2}) \longrightarrow (\Sigma_g, \sigma_{\Sigma_g})$, where $\lambda \geq 1$ and the two forms σ_{S^2} and σ_{Σ_g} have total area 1. Accordingly, we let the symplectomorphism groups G_λ^g be $\mathrm{Symp}(\Sigma_g \times S^2, \lambda\sigma_{\Sigma_g} \oplus \sigma_{S^2}) \cap \mathrm{Diff}_0(M_\lambda^g)$.

4.1 Prior results on the additive structure of $\pi_* G_\lambda^g$

We present here results that are essentially contained in McDuff [16]. Let us denote by $\mathcal{S}_\lambda^g$ the space of symplectic forms that are strongly isotopic with $\omega_\lambda := \lambda\sigma_{\Sigma_g} \oplus \sigma_{S^2}$, and by $\mathcal{A}_\lambda^g$ the space of almost complex structures that are tamed by some form in $\mathcal{S}_\lambda^g$. Then, there exists a fibration $G_\lambda^g \longrightarrow \mathrm{Diff}_0(M_\lambda^g) \longrightarrow \mathcal{S}_\lambda^g$ and, since $\mathcal{S}_\lambda^g$ is homotopy equivalent with $\mathcal{A}_\lambda^g$, there is also a homotopy fibration

$$G_\lambda^g \longrightarrow \mathrm{Diff}_0(M_\lambda^g) \longrightarrow \mathcal{A}_\lambda^g. \tag{19}$$

Let $D_k = A - kF \in H_2(M^g_\lambda, \mathbb{Z})$ where A and F are the homology classes of the base and the fiber, respectively. The subsets $\mathcal{A}^g_{\lambda,k}$ of $\mathcal{A}^g_\lambda$ consisting of almost complex structures that admit J-holomorphic curves in the class D_k provide a stratification of $\mathcal{A}^g_\lambda$ as in the following:

Proposition 4.1. (McDuff [16])

- **(i)** $\mathcal{A}^g_\lambda \subset \mathcal{A}^g_{\lambda+\varepsilon}$ and hence, via (19) one obtains maps $h_{\lambda,\lambda+\varepsilon} : G^g_\lambda \longrightarrow G^g_{\lambda+\varepsilon}$.
- **(ii)** $\mathcal{A}^g_{\lambda,k}$ is a Frechet suborbifold of $\mathcal{A}^g_\lambda$ of codimension $4k - 2 + 2g$.
- **(iii)** $\mathcal{A}^0_\lambda$ is constant on all the intervals $(\ell, \ell + 1]$ and $\mathcal{A}^0_{k+\varepsilon} \setminus \mathcal{A}^0_k = \mathcal{A}^0_{k+\varepsilon,k}$.
- **(iv)** The homotopy type of G^0_λ is constant for $k < \lambda \leq k+1$, with k an integer greater than zero. For this range of λ, there exists a nontrivial fragile element $w_k \in \pi_{4k}(G^0_\lambda) \otimes \mathbb{Q}$ that disappears when λ passes the critical value $k+1$, while a new fragile element w_{k+1} appears.
- **(v)** There exists a fragile element $\rho \in \pi_2(G^1_1)$ that disappears in $\pi_2(G^1_{1+\varepsilon})$. □

Moreover, the inclusions $i : G^g_\lambda \longrightarrow \mathrm{Diff}_0(M^g)$ lift to maps $\tilde{i} : G^g_\lambda \longrightarrow \mathcal{D}^g_0$ where $\mathcal{D}^g_0$ is the subgroup of diffeomorphisms that preserve the S^2 fibers. The following proposition shows that all essential elements in $\pi_*(\mathcal{D}^g_0)$ are retained in the homotopy groups of symplectomorphism groups:

Proposition 4.2. (McDuff [16])

- **(i)** The vector space $\pi_i(\mathcal{D}^g_0) \otimes \mathbb{Q}$ has dimension one when $i = 1, 3$ except in the cases $i = g = 1$ when the dimension is three, and $g = 0, i = 3$ when the dimension is two. It has dimension $2g$ when $i = 2$ and is zero otherwise.
- **(ii)** There exist maps $\tilde{i} : G^g_\lambda \longrightarrow \mathcal{D}^g_0$ that induce a surjection on all rational homotopy groups for all $g > 0$ and $\lambda \geq 0$. The map is actually an isomorphism on π_i, $i = 1, \ldots, 2g-1$ when we restrict to the range $\lambda > k$ where $g = 2k$ or $g = 2k+1$ depending on the parity.
- **(iii)** The map $\tilde{i}$ also gives an isomorphism on π_i for $g = 1, i = 2, 3, 4, 5$, and $\lambda > 3/2$.
- **(iv)** (Abreu–McDuff [3]) The homotopy limit $G^g_\infty = \lim_{\lambda \longrightarrow \infty} G^g_\lambda \approx \mathcal{D}^g_0$. □

4.2 Lie group actions on symplectic ruled surfaces

We will describe first all possible Hamiltonian circle actions on the manifolds M_λ^g. This appears for instance in M. Audin [9]. For these actions, we give a complete description of the equivariant Gromov–Witten invariants that count isolated curves of genus g. We also describe families of robust elements as in the hypothesis of Proposition 2.1 which, combined with the nontrivial count of EGW, yields nontrivial Whitehead products.

The Lie groups $H_k \approx S^1$ act on the manifolds M_λ^g, $\lambda > k$ as follows.

We denote by $\mathcal{O}(-2k)_g$ a holomorphic line bundle of degree $-2k$ over the surface Σ_g, and consider the projectivized line bundles $\pi : P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g) \longrightarrow \Sigma_g$. The Kähler manifolds $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ are endowed with naturally integrable almost complex structures denoted by $J^{(k),g}$. Topologically, they are just $\Sigma_g \times S^2$ and it is easy to see that these bundles admit a *holomorphic* circle action that rotates the fibers while fixing the zero section and the section at infinity that represents the classes $A - kF$ and $A + kF$, respectively:

$$\theta_k^g : S^1 \times P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g) \longrightarrow P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g). \tag{20}$$

In coordinates, this action is given by $e^{it} \cdot (b, [v_1 : v_2]) = (b, [e^{it} v_1 : v_2])$. We will view $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ as the symplectic manifolds M_λ^g endowed with the S^1-invariant taming complex structures $J^{(k),g}$ whenever $\lambda > k$.

The $J^{(k),g}$-holomorphic circle actions (20) become *Hamiltonian* with respect to the tamed symplectic forms ω_λ whenever $\lambda > k$; this is for example explained in [9]. The ruled surfaces M_λ^g, for $\lambda > 0$ can be constructed via symplectic reduction from disk bundles $D_a(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$ with appropriate radii a.

It is clear that the actions (20) cease to be symplectic whenever $\lambda \leq k$. However, for $\lambda > k$ they yield symplectic actions:

$$\theta_{\lambda,k}^g : S^1 \times M_\lambda^g \longrightarrow M_\lambda^g, \lambda > k. \tag{21}$$

To stress this distinction, we will use the notation $\gamma_{k,\lambda}^g : S^1 \longrightarrow G_\lambda^g$ for the group homomorphisms given by the symplectic actions (21). Also from McDuff's results that led to Proposition 4.2, we know that the cycles $\tilde{i}(\gamma_{\lambda,k}^g)$ are essential in $\mathcal{D}_0^g$ and represent an element in $\pi_1(\mathcal{D}_0^g) \otimes \mathbb{Q}$.

In fact, a smooth representative for this element can be given by maps $t_k^g : S^1 \longrightarrow \mathcal{D}_0^g$, with $[t_k^g] \in \pi_1(\mathcal{D}_0)^g \otimes \mathbb{Q}$ described as follows:

$$t_k^g(e^{it})(z, w) = (z, R_{e^{it}}^{\rho(z)}(w)) \tag{22}$$

where $\rho : \Sigma_g \longrightarrow S^2$ is a covering map of degree k and $R_{e^{it}}^{\rho(z)}(w)$ rotates the fiber sphere in $S^2 \times S^2$ with an angle t about a point $\rho(z)$ in the base sphere. So we have that $[t_k^g] = \tilde{i}_*[\gamma_{k,\lambda}^g]$.

4.2.1 *Toric actions K_k in the rational case*

In the case $g = 0$, the Hamiltonian H_k circle actions (20) are in fact induced from a T^2 *toric* action. Then, $M_\lambda^0, \lambda \geq 1$ can be obtained through symplectic reduction in $\lfloor \lambda \rfloor$ different ways as $M_\lambda^0 = \mathbb{C}^4 // T^2$, for any $0 \leq k < \lambda$ where the two generators ξ_1, ξ_2 of T^2 act on $\mathbb{C}^4$ with weights $(1, 1, 0, 0)$ and $(2k, 0, 1, 1)$. The subgroups K_k of symplectic automorphisms that commute with the toric action are compact Lie subgroups given by $K_k = S^1 \times SO(3)$ for $k > 0$ and $K_0 = SO(3) \times SO(3)$. The latter has two 2-torsion generators τ and τ' of $\pi_1 K_0$ and $\alpha := \alpha_0$ and η as two nontorsion generators in $\pi_3 K_0$. K_k has α_k as a nontorsion generator in $\pi_3 K_0$. We have that $H_k \subset K_k$.

Moreover, the inclusion $K_k \longrightarrow G_\lambda^0$ induces an injection on the rational homotopy groups. Since for $\lambda > k > 0$, $\pi_* K_k \otimes \mathbb{Q}$ has one generator γ_k in dimension one and one generator α_k in dimension three, while $\pi_3 K_0 \otimes \mathbb{Q}$ has two generators η and $\alpha_0 = \alpha$, it follows that $\pi_* G_\lambda^0 \otimes \mathbb{Q}$ has one generator $\gamma_{\lambda,k}^0$ in dimension one when $\lambda > k > 1$, and two generators α, β in dimension three when $\lambda > 1$.

The following lemma, essentially proved in [3] gives us relations between the nontrivial images of these generators of H_k and K_k when viewed inside an appropriate G_λ.

Lemma 4.3.

(i) Consider (1) $g > 0$ and $k \geq 1$, or (2) $g = 0$ and $k \geq 2$.

$$[t_k^g] = k[t_1^g] \in \pi_1 \mathcal{D}_0^g \otimes \mathbb{Q}. \tag{23}$$

(ii) If, in addition, we assume $\lambda > k > [g/2]$, then the same relation takes place between the symplectic representatives in $\pi_1 G^g_\lambda \otimes \mathbb{Q}$:

$$[\gamma^g_{\lambda,k}] = k[\gamma^g_{\lambda,1}] \in \pi_1 G^g_\lambda \otimes \mathbb{Q}. \tag{24}$$

(iii) For $g = 0$, we have

$$\alpha_k = \alpha_0 + k^2\eta \in H_3(G^0_\lambda, \mathbb{Q}). \tag{25}$$

(iv) There exists a continuous family of robust elements of infinite order

$$\delta^g_{\lambda,k} : S^1 \longrightarrow G^g_\lambda, \lambda \geq k, \tag{26}$$

such that for $\lambda > k$ the elements $\delta^g_{\lambda,k}$ are homotopic with the homomorphisms $\gamma^g_{\lambda,k}$ given by the group action H_k.

Moreover, if $\lambda \geq k > [g/2]$ when $g > 1$ or for any λ when $g = 0$, at the critical values $\lambda = k$ we have integers M so that

$$[\delta^g_{k,k}] = M[\gamma^g_{k,1}] \in \pi_1 G^g_k \otimes \mathbb{Q}. \tag{27}$$

□

Proof. The proof of (i) is an immediate adaptation of Lemma 2.10 proved in Abreu–McDuff [3] for the case $g = 0$. In fact they actually compute the difference between the two terms as a 2-torsion element. Similarly, when we restrict to the given range for λ the morphisms $\tilde{i}$ give an isomorphism on π_1 and hence the relation in (i) continues to hold in $\pi_1 G^g_\lambda \otimes \mathbb{Q}$. Part (iii) is also contained in Lemma 2.10 proved in Abreu–McDuff [3].

The existence of the robust family (26) in part (iv) is an immediate consequence of Proposition 4.2. Indeed, since the maps $\tilde{i}$ induce a surjection on the first rational homotopy groups for λ in that range and the family can be obtained by pulling back the smooth representative t^g_k to the symplectomorphism groups.

For the second part of (iv), we use the fact that $\tilde{i}$ induces an isomorphism on the first rational homotopy groups for the given λ range. Using this isomorphism, the relation (27) follows for instance from the fact that for the given λ range the vector space $\pi_1 G^g_k \otimes \mathbb{Q}$ is one dimensional (Proposition 4.2 point (i)). ■

Of particular importance for us are the relations in point (i). For the rational case, dual relations between the elements in the cohomology, derived by Abreu–Granja–Kitchloo [2] will be provided and used in Section 4.6.

4.3 Natural Equivariant Gromov–Witten invariants in ruled surfaces

The circle maps

$$\gamma^{g}_{\lambda,k} : S^1 \longrightarrow \mathrm{Symp}(M^g_\lambda) \tag{28}$$

give rise to maps

$$\mathrm{B}\gamma^{g}_{\lambda,k} : \mathrm{B}S^1 \longrightarrow \mathrm{BSymp}(M^g_\lambda). \tag{29}$$

These can also be viewed as a collection of symplectic fibrations

$$(Q_\lambda^{(k),(p),g}, J^{(k),(p),g}) = M^g_\lambda \times_{H_k} S^{2p+1}. \tag{30}$$

They are the associated symplectic fibrations with fiber $(M^g_\lambda, \omega_\lambda)$ endowed with the S^1-invariant symplectic form $\Lambda_\lambda^{(k),(p)}$ and compatible almost complex structure $J^{(k),(p),g}$.

We will show in 4.8 that $\mathrm{EGW}^{(p)}_{g,0}(Q^{(p),g}_\lambda, s_{D_k}) = \pm 1$ if $p = 2k+g-1$ and zero otherwise. That allows us to state the following:

Theorem 4.4. For any arbitrary genus g, and a Hamiltonian circle action with Lie group H_k on M^g_λ, $\lambda > k$ as in (21), and equivariant almost complex structure $J^{(k),g}$, we have

$$\mathrm{EGW}_{g,0}(M^g_\lambda; H_k; s_{A-kF}) = \pm 1 \cdot u^{2k+g-1} \in H^*(\mathrm{B}S^1, \mathbb{Q}). \qquad \square$$

Then, using Corollary 3.11 we obtain the following

Corollary 4.5. The fibrations $(Q_\lambda^{(k),(p),g}, J^{(k),(p),g})$ that have nontrivial EGW provide us with a family of maps:

$$\widetilde{F}^g_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow \mathrm{B}G^g_{k+\varepsilon},\ \varepsilon > 0$$

with $n+1 = 2k+g-1$. This family and its multiples are **new families** as in condition I of Corollary 2.6. $\square$

The proof follows from Theorem 1.6 and Corollary 3.11. This is the chief result that will be used in the following three subsections.

4.4 The case $g = 1$

Here we prove Theorem 1.4. When $k = 1$ and $g = 1$, Corollary 4.5 gives a family $\widetilde{F}^g_{2,\varepsilon}$: $\mathbb{P}^2 \longrightarrow \mathrm{B}G^1_{1+\varepsilon}$ ($\varepsilon > 0$). Consider the family $\delta^1_{1+\varepsilon}$ given by (26). For simplicity, we have suppressed the double subscript. The first element of the family is an essential loop

$$\delta^1_1 : S^1 \longrightarrow G^1_1.$$

By suspension of the family $\delta^1_{1+\varepsilon}$, we get maps $F_{1,\varepsilon} : \mathbb{P}^1 \longrightarrow \mathrm{B}G^1_{1+\varepsilon}$, $\varepsilon \geq 0$. Assume that the Whitehead product $[F_1, F_1] = 0$. Then, a multiple of F_1 extends to F_2 and further deforms to $F_{2,\varepsilon}$. Lemma 4.3 (iv) implies that suitable multiples of $F_{1,\varepsilon}$ and $\widetilde{F}_{1,\varepsilon}$ will be homotopic for $\varepsilon > 0$. Since $\widetilde{F}_{2,\varepsilon}$ is a new family and $F_{2,\varepsilon}$ is a deformation of the map F_2, the two cannot be homotopic; therefore we must have a new element in $\pi_4(\mathrm{B}G^1_{1+\varepsilon})$. We claim that this element is in fact robust. For this, recall that $\tilde{F}_{2,\varepsilon}$ exhibits nontrivial Gromov–Witten invariants in a class on which the symplectic form ω_1 vanishes. If the two fibrations $\tilde{F}_{2,\varepsilon}$ and $F_{2,\varepsilon}$ become homotopic after a large ε, they would have the same (nontrivial) Gromov–Witten invariants since these invariants are unchanged under deformations. Recall that F_2 represents a symplectic fibration whose vertical two form restricts in fibers to symplectic forms in the class $[\omega_1]$. Hence, F_2 would have a nontrivial Gromov–Witten invariant in a homology class on which its underlying symplectic form would have to vanish, which is impossible.

Therefore, this robust element must survive in the homotopic limit $\mathrm{B}G^1_\infty = \mathrm{B}\mathcal{D}^1_0$. But since the maps $\widetilde{i}$ are surjective on rational homotopy groups for any $1 + \varepsilon \geq 1$, no essential element in $\mathrm{B}\mathcal{D}^1_0$ can be obtained from a new element detected after some $1 + \varepsilon > 1$. Therefore, our assumption that the Whitehead product is trivial must be false and Theorem 1.4 holds.

Remark 4.6. We know from Karshon's results in [13] that no topologically essential circle map $S^1 \longrightarrow G^1_1$ can be realized as a group homomorphism. In particular, $\delta^1_1 : S^1 \longrightarrow G^1_1$ is not an A_∞ map. We have shown that the first homotopic obstruction to realize δ^1_1 as an H-homomorphism is nontrivial and hence δ^1_1 cannot even be realized as A_1 map. □

4.5 The case $g = 0$

Corollary 4.5 gives a family $\widetilde{F}_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow \mathrm{B}G^0_{k+\varepsilon}$, $\varepsilon > 0$ for $n = 2k-2$. This family and its multiples are new. Use now $\mathrm{B}\gamma^0_{k,1} : \mathrm{B}S^1 \longrightarrow \mathrm{B}G^0_k$ to get a map $F_1 : \mathbb{P}^1 \longrightarrow \mathrm{B}G^0_k$.

Recall that the maps $h_{k,k+\varepsilon} : \mathrm{B}G^0_k \longrightarrow \mathrm{B}G^0_{k+\varepsilon}$ induce an isomorphism on $\pi_*(\mathrm{B}G^0_{k+\varepsilon})$ for $* \leq 2n = 4k-4$.

By composition, we obtain a family

$$F_{1,\varepsilon} = h_{k,k+\varepsilon} \circ F_1 : \mathbb{P}^1 \longrightarrow \mathrm{B}G^0_{k+\varepsilon}, \quad \varepsilon \geq 0.$$

Observe that after considering sufficient multiples for both $F_{1,\varepsilon}$ and $\widetilde{F}_{n+1,\varepsilon}$, we get via Lemma 4.3 that the maps $F_{1,\varepsilon}$ and $\widetilde{F}_{1,\varepsilon}$ are homotopic to each other, have no torsion, and are homotopic to a multiple of the suspension of the circle map $\gamma^0_{k+\varepsilon,1}$.

We want to build on F_1 a map F_n whose deformations via composition with $h_{k,\varepsilon}$ are homotopic to $\widetilde{F}_{n,\varepsilon}$. Note that the homotopy type of the map $\widetilde{F}_{n,\varepsilon}$ is determined by all the homotopy classes of the attaching maps used to attach subsequent cells to $\widetilde{F}_{1,\varepsilon}$. Since the spaces $\mathrm{B}G^0_k$ and $\mathrm{B}G^0_{k+\varepsilon}$ have isomorphic π_* for $* \leq 2n$, and the isomorphism is preserved by the maps $h_{k,\varepsilon}$, the same attaching maps used for $\tilde{F}_{n,\varepsilon}$ can be used to extend F_1 to the desired map $F_n : \mathbb{P}^n \longrightarrow \mathrm{B}G^0_k$.

Evidently, by this construction, the deformations

$$F_{n,\varepsilon} = h_{k,\varepsilon} \circ F_n : \mathbb{P}^n \longrightarrow \mathrm{B}G^0_{k+\varepsilon}, \quad \varepsilon \geq 0.$$

are homotopic to $\widetilde{F}_{n,\varepsilon}$ for $\varepsilon > 0$.

It follows immediately that as long as $\varepsilon > 0$, we can extend $F_{n,\varepsilon}$ to a map $F_{n+1,\varepsilon}$ homotopic to $\widetilde{F}_{n+1,\varepsilon}$. The family $F_{n+1,\varepsilon}$, $\varepsilon > 0$ satisfies conditions I and II of Corollary 2.6. Hence, either point A) or B) must hold. An argument as in Subsection 4.4 shows that any elements obtained by point A must be robust.

However, we cannot have any new robust families in $\pi_{2n+2}(\mathrm{B}G^0_{k+\varepsilon})$ since any such element would vanish after reaching the appropriate integer. Hence, B) holds and we must have a nonzero element $W \in \pi_{2n+1}\mathrm{B}G^0_k \otimes \mathbb{Q}$ that gives a nontrivial Whitehead product of order $2n$. Recall that we considered $n = 2k-2$. Now, we have the following:

Proposition 4.7. For all $k \geq 1$ and $0 < \varepsilon \leq 1$, and for $\gamma^0_{k+\varepsilon,1}$ given by the circle action H_k, the rational Samelson product of order $2k+1$, $S^{(2k+1)}(\gamma_{k+\varepsilon,1})$, is equal to $\{0, w_k\} \subset \pi_{4k}(G_{k+\varepsilon}) \otimes \mathbb{Q}$, where w_k is a fragile homotopy class that disappears when $\varepsilon > 1$. □

Proof. Recall that an nth-order Samelson product is a desuspension of a corresponding Whitehead product. The previous arguments, shifted from k to $k+1$, show that for $w_k \in \pi_{4k}(G_{k+1})$ we have $\{w_k\} \subset S^{(2k+1)}(\gamma^0_{k+1,1})$. Zero is always present in the rational Samelson products of $S^{(2k+1)}(\gamma_{1,k+1})$ because the map $\gamma^0_{k+1,1}$ is a symplectic circle action. Furthermore, since the rational homotopy groups of G_{k+1} are known, no other elements will be present in this Samelson product and the lower-order products must be trivial. Finally, since the groups $G_{k+\varepsilon}$ are homotopic for $0 < \varepsilon \leq 1$ the proposition holds. ■

4.6 The ring $H^*(\mathrm{B}G^0_\lambda; \mathbb{Q})$ and relations with Abreu–Granja–Kitchloo's work

4.6.1 *Minimal rational Whitehead order of* $\mathrm{B}G_\lambda$ *and consequences to its minimal model*

We aim here to understand all the rational Whitehead products among elements in the homotopy groups $\pi_*\mathrm{B}G_\lambda \otimes \mathbb{Q}$. We will write A for the suspension of $\gamma^0_{\lambda,1}$ in $\pi_2\mathrm{B}G^0_\lambda \otimes \mathbb{Q}$, X for the suspension of α_0 in $\pi_4\mathrm{B}G^0_\lambda \otimes \mathbb{Q}$, and Y for the suspension of η in $\pi_4\mathrm{B}G^0_\lambda \otimes \mathbb{Q}$. W_k represents the suspension of w_k in $\pi_{4k+1}\mathrm{B}G^0_\lambda \otimes \mathbb{Q}$.

Recall that $r \geq 2$ is the *the rational minimal Whitehead order* of a topological space B if it is the minimal order in which there exists a nonvanishing rational Whitehead product. Due to a result of Andrews–Arkovitz [6] if each homotopy group $\pi_*B_\emptyset$ of the rationalization of a space B is finitely generated then any rational Whitehead product of minimal order r contains exactly one element.

We will call Whitehead products of type (p, s), $2p + 4s = 4k + 2$ the following:

$$W^{(p,s)}(A, a_i X + b_i Y)_{i=\overline{1,k}} = [A, \ldots, A, a_1 Y + b_1 X, \ldots, a_s Y + b_s X]. \tag{31}$$

Lemma 4.8. For any $k \geq 1$ and $k < \lambda \leq k+1$, we have

1. Any Whitehead product of order less than $k+1$ vanishes. Moreover, all Whitehead products of order $k+1$ of type $(1, k)$ with $a_i = 1, b_i = d^2, 1 \leq d \leq k$ also vanish.
2. The following Whitehead product of order $k+1$ is nontrivial and consists of only one element in $\pi_{4k+1}\mathrm{B}G^0_\lambda$:

$$0 \neq [A, Y, \ldots, Y]. \tag{32}$$

□

Proof. Clearly, $[A, A] = 0$. Considerations of the dimension of $\pi_* \mathrm{B}G_\lambda$ imply that any other Whitehead products of order strictly less than $k+1$ must also vanish. Hence, any Whitehead product of order $k+1$ is defined and contains only one element. The classifying spaces of Lie subgroups K_i, $i \le k$ of G^0_λ are rationally H-space and hence they have vanishing rational Whitehead products. Lemma 4.3 (ii) and (iii) yield part (i) of the present lemma.

For the second part let us first notice that the indeterminacy in the Whitehead product $W^{(2k+1)}(A)$ obtained in Proposition 4.7 implies that nonvanishing lower-order Whitehead products must exist. Again, due to dimension considerations, they can only be of order (p, s) with $p+s < 2k+1$ and $2p+4s = 4k+2$. We use the following: ■

Lemma 4.9. The minimum Whitehead order is $k+1$.

The proof is by contradiction. Assume that the minimum Whitehead order is $p+s > k+1$. Hence, $p > 1$ and as above $2p+4s = 4k+2$. The following equation in b has degree s and coefficients in $\pi_{4k+1}\mathrm{B}G_\lambda \otimes \mathbb{Q}$ given by single element Whitehead products of type (p, s) generating all the Whitehead products of type (p, s):

$$0 = [A, \ldots, A, Y + bX, \ldots, Y + bX]. \tag{33}$$

Proposition 4.3 implies that Equation (33) has k solutions $b = 1, 4, \ldots k^2$ provided by the k different Lie group actions. If $p > 1$, then $k = \frac{2p+4s-2}{4} > s$ and hence all the coefficients must be zero. But they generate all Whitehead products of the given type (p, s), therefore p must be 1.

Lemma 4.9 and part (1) of Lemma 4.8 yield part (2) of Lemma 4.8. □

Proposition 4.10. Consider $k < \lambda \le k+1$. Consider the Sullivan minimal model $\mathcal{M}$ of $H^*(\mathrm{B}G^0_\lambda, \mathbb{Q})$.

- $\mathcal{M}$ is a commutative differential graded algebra $(\mathcal{M}, d)$ generated by $\overline{A} \in \mathcal{M}^2$, $\overline{X}, \overline{Y} \in \mathcal{M}^4$, and $\overline{W_k} \in \mathcal{M}^{4k+1}$ dual to A, X, Y, W_k.
- We have the following isomorphisms of graded rings

$$H^*(\mathrm{B}G_\lambda; \mathbb{Q}) = H^*_{\text{cochain}}(\mathcal{M}; \mathbb{Q}) = \mathbb{Q}[\overline{A}, \overline{X}, \overline{Y}]/\langle F'_k \rangle \tag{34}$$

where $\overline{A}, \overline{X}, \overline{Y}$ have degrees 2, 4, and 4 and $\langle F_k' \rangle$ is the principal ideal generated by a polynomial F_k' of homogeneous degree $4k+2$. □

Proof. A complete set of generators for the Sullivan minimal model's differential algebra $\mathcal{M}$ of $\mathrm{B}G_\lambda^0$ is given by elements in the dual homotopy groups $\mathrm{Hom}(\pi_* \mathrm{B}G_\lambda^0 \otimes \mathbb{Q}, \mathbb{Q})$, in our case consisting of $\overline{A} \in \mathcal{M}^2$, $\overline{X}, \overline{Y} \in \mathcal{M}^4$, and $\overline{W_k} \in \mathcal{M}^{4k+1}$, dual to A, X, Y, and W_k. Degree considerations insure that $\mathcal{M}^{2i+1} = 0$ for $2i+1 < 4k+1$. Therefore, $d\overline{A} = d\overline{X} = d\overline{Y} = 0$ and these generators transgress to the cochain cohomology on $\mathcal{M}$. Also due to degree considerations, we have that $\mathcal{M}^{4k+2}$ must be generated by homogeneous monomials in $\overline{A}, \overline{X}, \overline{Y}$ of degree $4k+2$. Thus, $d\overline{W_k} = F_k'$, for F_k' some homogeneous polynomial of degree $4k+2$.

Any complete set of generators on $\mathcal{M}$ induces a decreasing filtration $\mathcal{M}_s^\circ$ on the quotient of $\mathcal{M}$ by the elements of degree 0 denoted by$\mathcal{M}^\circ$, with $\mathcal{M}_s^\circ$ being the subalgebra generated by products of s generators.

Since the Whitehead minimal order is $r = k+1$, [6, Proposition 6.4] implies that for any $\mu \in \mathcal{M}$ with nontrivial differential we must have $d\mu \in \mathcal{M}_{k+1}^\circ$. Moreover, Theorem 5.4 in [6] states that for any μ with $d\mu \in \mathcal{M}_s^\circ$, and $z \in [x_1, x_2, \ldots, x_s] \in \pi_*(\mathrm{B}G_\lambda^0) \otimes \mathbb{Q}$, the (partial) differential $d\mu$ modulo $\mathcal{M}_{k+2}^\circ$ can be computed via Sullivan pairings $\langle \bar{\mu}, z \rangle$. In our case that yields the nontrivial part of the polynomial F_k' consisting of homogeneous monomials of exact word length $k+1$. Leibnitz rule implies that any other nontrivial differential of higher degree would be divisible by $\overline{W_k}$. Hence, $H^*_{\mathrm{cochain}}(\mathcal{M}; \mathbb{Q}) = \mathbb{Q}[\overline{A}, \overline{X}, \overline{Y}] / < F_k' >$. ■

4.6.2 *The cohomology ring* $H^*(\mathrm{B}G_\lambda; \mathbb{Q})$

We will compute here the full cohomology ring $H^*(\mathrm{B}G_\lambda; \mathbb{Q})$. This result was proved using different methods in [2]. In [3], Abreu–McDuff obtained nontriviality of the Whitehead products by different means and obtained partial relations. We use the algebraic computations from [2] that explain how the cohomology generators of $H^*(\mathrm{B}G_\lambda; \mathbb{Q})$ restrict to the cohomologies $H^*(\mathrm{B}K_i; \mathbb{Q})$. These, combined with Proposition 4.10, will yield the result.

In order to match our computations to those in [2], we set $\overline{Z}$ to be

$$\overline{Z} = \overline{Y} + \overline{A}^2. \tag{35}$$

Denote by $P_i = i^4\overline{X} - i^2\overline{Z} + \overline{A}^2$. The following algebraic computation from [Corollary 5.16] from [2] gives us a description of the cohomologies $H^*(\mathrm{B}K_i; \mathbb{Q})$:

Proposition 4.11. (Abreu–Granja–Kitchloo) [2] If $i > 0$, then $H^*(\mathrm{B}K_i; \mathbb{Q}) = \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle P_i\rangle$. □

Note that with the above variables one is able to draw the same conclusion on the rational cohomology of $\mathrm{B}G_\lambda$ as in Proposition 4.10. We can now provide an alternate proof for Abreu–Granja–Kitchloo's Theorem 1.8:

Proof of Theorem 1.8. By Proposition 4.10, $H^*(\mathrm{B}G^0_\lambda, \mathbb{Q}) = Q(\overline{A}, \overline{X}, \overline{Z})/\langle F_k\rangle$, with F_k of homogeneous degree $4k+2$. We will show by induction on k that $F_k = \overline{A}P_1 \cdots P_k$. Assume that $F_{k-1} = \overline{A}P_1 \dots P_{k-1}$. The maps $f_\lambda := \mathrm{B}h_{\lambda-1,\lambda} : \mathrm{B}G_{\lambda-1} \longrightarrow \mathrm{B}G_\lambda$ and $\xi_k : \mathrm{B}K_k \longrightarrow \mathrm{B}G_\lambda$ induce morphisms of graded rings between the corresponding rational cohomology rings. By Proposition 4.10 and Proposition 4.11, they give the graded ring morphisms:

$$f^*_\lambda : \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle F_k\rangle \longrightarrow \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle F_{k-1}\rangle, \tag{36}$$

$$\xi^*_k : \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle F_k\rangle \longrightarrow \mathbb{Q}[\overline{A}, \overline{X}, \overline{Z}]/\langle P_k\rangle. \tag{37}$$

These relations imply that both F_{k-1} and P_k must divide F_k. By Proposition 4.10, the homogeneous degree of F_k must be $4k+2$. Since all factors $\overline{A}$, P_i, $1 \le i \le k-1$ in F_{k-1} are mutually prime with P_k, it follows that F_k is of the form (1).

The initial of induction follows similarly; in this case $P_1 = \overline{X} - \overline{Z} + \overline{A}^2$ divides F_1 from the map (37) and $\overline{A}$ divides F_1 from the effect in cohomology of the inclusion of K_0 in G_λ, $1 < \lambda \le 2$. Since F_1 has homogeneous degree 6, it immediately follows that $F_1 = \overline{A}P_1$. ■

4.7 The case $g > 1$

We will treat this case in a manner similar to the case $g = 0$, but since a lot less is known about the additive structure of $\pi_*(\mathrm{B}G^g_{k+\varepsilon})$ in this case, our conclusions will be weaker.

Proposition 4.12. For all genus $g \ge 2$ and all $k > [g/2]$, there exist elements $\gamma^g \in \pi_1 G^g_k \otimes \mathbb{Q}$ with nonvanishing Samelson product of order r with $g \le r \le 2k+g-1$, $0 \ne w^g_r \in S^{(r)}(\gamma^g) \subset \pi_{2r-2}(G^g_k)$. □

Proof. Corollary 4.5 provides a new family $\widetilde{F}^g_{n+1,\varepsilon} : \mathbb{P}^{n+1} \longrightarrow \mathrm{B}G^g_{k+\varepsilon}$, $n+1 = 2k+g-1$. These maps are just restrictions of $\mathrm{B}\gamma^g_{k+\varepsilon,k} : \mathrm{B}S^1 \longrightarrow \mathrm{B}G^g_{k+\varepsilon}$ to the $2n+2$ skeleton.

As in the rational case, we get $F_1 : \mathbb{P}^2 \longrightarrow \mathrm{B}G^g_k$ as the suspension of $\gamma^g_{k,1}$. Again, after taking suitable multiples of F_1 and $\widetilde{F}_{n+1,\varepsilon}$ we can assume that the deformations $F_{1,\varepsilon}$ and $\widetilde{F}_{1,\varepsilon}$ are homotopic when $\varepsilon > 0$. We will now restrict ourselves to the case when $k > [g/2]$. Since the suspension of the map $\widetilde{i}$ gives an isomorphism π_*, $* \leq 2g$, we can import the suspensions of the classes the attaching maps that used to build the map $\widetilde{F}_{g,\varepsilon}$ in $\mathrm{B}G^g_k$. That way we get an extension $F_g : \mathbb{P}^g \longrightarrow \mathrm{B}G^g_k$ whose deformations $F_{g,\varepsilon}$ obtained via composition with the maps $h_{k,k+\varepsilon}$ are homotopic to $\widetilde{F}_{g,\varepsilon}$. Hence for $\varepsilon > 0$, we can homotope the maps $\widetilde{F}_{n+1,\varepsilon}$ to maps $F_{n+1,\varepsilon}$ whose restrictions to the $2g$ skeleton coincide with the maps $F_{g,\varepsilon}$ obtained by the above procedure.

The family $F_{n+1,\varepsilon}$ satisfies Corollary 2.7 with $k = g$. From the conclusion of the corollary, either C) or D) must hold.

Assume that C) happens. Then there is an r with $k \leq r \leq p$ and a new family $\eta_\varepsilon : S^{2r+2} \longrightarrow \mathrm{B}G^g_{k+\varepsilon}$. This family exists for any $\varepsilon > 0$ but it might become null-homotopic at some ε_0. When that happens it means that the fibrations $F_{r,k+\varepsilon_0}$ and $\widetilde{F}_{r,k+\varepsilon_0}$ become homotopic. Then we apply Corollary 2.7 again for $\lambda_0 = k + \varepsilon_0$ and $k = r$. If situation D) never happens, and if all the new families η_ε found by this process become null-homotopic after finite time, we can conclude that there is some ε' such that the deformations $F_{n+1,\varepsilon'}$ of extensions of F_g are homotopic to $\widetilde{F}_{n+1,\varepsilon'}$. But this cannot happen since the latter has nontrivial PGW in a homology class on which the symplectic form ω^g_k vanishes.

Therefore, if D) never holds we must have robust new elements $0 \neq \eta_\varepsilon \in \pi_r \mathrm{B}G^g_{k+\varepsilon} \otimes \mathbb{Q}$, $\varepsilon \longrightarrow \infty$. Any such element would have to survive in the homotopic limit $\mathrm{B}G^g_\infty = \mathrm{BD}^g_0$, which is impossible as the rational homotopy of $\mathrm{B}\mathcal{D}^g_0$ is known from 4.2 part (i). So D) must hold and the proposition follows. ■

Corollary 4.13. For all genus $g \geq 2$ and all $k > [g/2]$, the groups $\mathrm{B}G^g_k$ do not have the homotopy type of a compact Lie group. □

Proof. Since the classifying spaces $\mathrm{B}G^g_k$ have nontrivial rational Whitehead products, they cannot be H-spaces. ■

4.8 Computation of equivariant Gromov–Witten invariants on ruled surfaces M^g_λ

Proof of Theorem 1.6. Recall that

$$(Q^{(k),(p),g}_\lambda, J^{(k),(p),g}) = M^g_\lambda \times_{H_k} S^{2p+1} \tag{38}$$

is the associated symplectic fibration with fiber $(M_\lambda^g, \omega_\lambda)$ endowed with the symplectic form $\Lambda_\lambda^{(k),(p)}$ and compatible almost complex structure $J^{(k),(p),g}$. Then according to Proposition 3.9, we need to show that $\mathrm{EGW}_{g,0}^{(p)}(Q_\lambda^{(p),g}, s_{D_k})$ is ± 1 if $p = 2k + g - 1$ and zero otherwise.

The dimension condition in (18) translates into saying that

$$(\dim_{\mathbb{C}} M_\lambda^g - 3)(1-g) + c_1(A - kF) + 2p = g - 1 + (A - kF)^2 + 2 - 2g + 2p$$
$$= g - 1 - 2k + 2 - 2g$$
$$= -2k - g + 1 + 2p \tag{39}$$

must be 0. Therefore, all such invariants are zero unless $p = 2k + g - 1$.

In this situation, there exists exactly one embedded vertical $J^{(k),(p),g}$-holomorphic map representing s_{D_k} in each fiber $Q_b^{(k),(p),g}$ for each $b \in \mathbb{P}^p$. More precisely, each fiber is biholomorphic to $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$. The only possible bubbling for vertical almost holomorphic curves in $Q_b^{(k),(p),g}$ must take place within a fiber. It immediately follows that the only $J^{(k),(p),g}$ maps in each fiber representing the class D_k is the zero section of the bundle $P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$. Therefore, the moduli space $\mathcal{M}_{g,0}(Q^{(k),(p),g}, J^{(k),(p),g}, s_{D_k})$ is naturally diffeomorphic with $\mathbb{P}^p$.

Given such $J^{(k),(p),g}$-holomorphic map $f : (\Sigma_g, j_g) \longrightarrow (M_\lambda^g, J^{(k),(p),g})$ in the class D_k, the linearized operator $D\phi$ of index zero is

$$D\phi([b, f, j_g]) : T_b\mathbb{P}^p \times C^\infty(f^*TM_\lambda^g) \times T_{j_g}\mathrm{Teich}_g \longrightarrow \Omega^{(0,1)}(f^*TM_\lambda^g) \tag{40}$$

where the component corresponding to the Teichmüller space appears when $g > 0$.

The actual dimension of $\mathcal{M}_{g,0}(Q^{(k),(p),g}, J^{(k),(p),g}, s_{D_k})$ is larger than its formal dimension zero. This is because the fiberwise almost complex structure $J^{(k),(p),g}$ is not D_k *regular*, or equivalently, the linearized operator (40) is not onto. The computation of the invariants then follows from the following:

Lemma 4.14.

(i) $\mathrm{EGW}_{g,0}^{(p)}(Q^{(k),(p),g}, s_{D_k}) = e(\mathcal{O}^g)$ where $e(\mathcal{O}^g)$ represents the Euler class of the obstruction bundle $\mathcal{O}^g \longrightarrow \mathcal{M}_{g,0}(Q^{(k),(p),g}, J^{(k),(p),g}, s_{D_k})$ induced by the section ϕ whose fiber over a point $[b, f, j_g]$ is given by $\mathrm{coker} D\phi([b, f, j_g])$.

(ii) Whenever $p = 2k + g - 1$, the obstruction bundle $\mathcal{O}^g \longrightarrow \mathcal{M}_{g,0}(Q^{(k),(p),g}, J^{(k),(p),g}, s_{D_k})$ is isomorphic to $\mathcal{O}_{\mathbb{C}P^p}(-1)^p \longrightarrow \mathbb{P}^p$. ■

Proof. (i) This follows immediately from the setup in the general theory as in Li–Tian [15], since in this particular case the moduli space $\phi^{-1}(0)$ is smooth and hence the generalized Fredholm orbifold is in fact a smooth vector bundle over $\mathbb{P}^p$.

(ii) Since f represents the zero section in the fiber $Q_b = P(\mathcal{O}(-2k)_g \oplus \mathcal{O}_g)$, the vertical tangent bundle $T_b^{\text{vert}}(Q_b^{(k)(p),g})_{|\text{im} f} = T(M_\lambda^g)_{|\text{im} f}$ splits holomorphically in the direct sum $T\Sigma_g \oplus \nu_g$, where ν_k^g is the normal bundle to the image Σ_g of the zero section f. It is immediate that the normal bundle is in fact $\mathcal{O}(-2k)_g \longrightarrow \Sigma_g$. The operator (40) becomes:

$$D\phi([b, f, j_g]) : T_b\mathbb{P}^p \oplus C^\infty(\Sigma_g, \nu_k^g) \oplus C^\infty(\Sigma_g, T\Sigma_g) \oplus T_{j_g}\text{Teich}_g \longrightarrow$$
$$\longrightarrow \Omega^{(0,1)}(\Sigma_g, \nu_k^g) \oplus \Omega^{(0,1)}(\Sigma_g, T\Sigma_g)$$

and hence

$$D\phi[b, f, j_g]) : T_b\mathbb{C}P^p \oplus C^\infty(\Sigma_g, \mathcal{O}(-2k)_g) \oplus C^\infty(\Sigma_g, T\Sigma_g) \oplus T_{j_g}\text{Teich}_g \longrightarrow$$
$$\longrightarrow \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g) \oplus \Omega^{(0,1)}(\Sigma_g, T\Sigma_g).$$

We will study the cokernel in the case $g = 0$ separately. If $g > 0$, then the component of $D\phi[b, f, j_g])$ that is not onto is

$$D\phi_{\text{restr}}([b, f, j_g]) : C^\infty(\Sigma_g, \mathcal{O}(-2k)_g) \longrightarrow \Omega^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g) \tag{41}$$

whose cokernel is $H^{(0,1)}(\Sigma_g, \mathcal{O}(-2k)_g)$. If we denote by K_g the degree $2g-2$ canonical bundle over Σ_g then, by Serre duality, $\text{coker}D\phi[b, f, j_g])$ will be precisely the space of holomorphic sections $(H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^*$. By the Riemann–Roch theorem, this space has complex dimension $2k + 2g - 2 - g + 1 = 2k + g - 1$. To find out how these fibers fit together topologically in the obstruction bundle, we need to understand what is the induced S^1-action on $(H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^*$ such that

$$\mathcal{O}^g = (H^0(\Sigma_g, \mathcal{O}(-2k)_g^* \otimes K_g))^* \times_{S^1} S^{2p+1}.$$

Since S^1 acts with weight 1 on the normal bundle $\mu_g = \mathcal{O}(-2k)_g$, and correspondingly on its dual $\mathcal{O}(-2k)_g^*$, the space of sections inherits a diagonal S^1-action with equal weights given by either 1 or -1. Since it will be enough to determine the EGW up to a sign, we will assume for simplicity that the weights are equal to 1. Since $\mathrm{im}(f)$ is a fixed set of the canonical bundle S^1-action, $T(Q^{(k),(p),g})_{|\mathrm{im}\, f}$ is also fixed by the induced S^1-action and therefore so is the K_g. Hence, the action on $(H^0(\Sigma_g, \mathcal{O}(-2k)^*_g \otimes K_g))^*$ is induced by the S^1-action with weights $(1, \ldots, 1)$ on $\mathcal{O}(-2k)^*_g$ and hence it is diagonal with weights $(1, \ldots, 1)$. It immediately follows that $(H^0(\Sigma_g, \mathcal{O}(-2k)^*_g \otimes K_g))^* \times_{S^1} S^{2p+1}$ is given by $\mathcal{O}_{\mathbb{P}^p}(-1)^p \longrightarrow \mathbb{P}^p$.

In the case $g = 0$, the moduli spaces involved in the computation must be of *unparametrized curves*, which means we have to quotient out the six-dimensional group $PGL(2, \mathbb{C})$ representing the reparametrizations of the domain. The linearized operator will be

$$D\phi([b, f, j_0]) : T_b\mathbb{P}^p \oplus C^\infty(\Sigma_0, \nu^0_k) \oplus C^\infty(S^2, TS^2) \longrightarrow \Omega^{(0,1)}(S^2, \nu^g_k) \oplus \Omega^{(0,1)}(S^2, TS^2)$$

with the cokernel given by:

$$D\phi_{\mathrm{restr}}([b, f, j]) : C^\infty(S^2, \mathcal{O}(-2k)) \longrightarrow \Omega^{(0,1)}(S^2, \mathcal{O}(-2k)).$$

A similar line of thought as above then applies. In this case, the canonical bundle is of negative degree $\mathcal{O}(-2)$ and the fiber of the obstruction bundle is $(H^0(S^2, \mathcal{O}(-2k)^* \otimes \mathcal{O}(-2))^* = (H^0(S^2, \mathcal{O}(2k-2))^*$ of complex dimension $2k - 1$.

Hence whenever $p = 2k + g - 1$, we have $\mathrm{EGW}^{(p)}_{g,0}(Q^{(k),(p),g}, s_{D_k}) = e(\mathcal{O}^g) = c_n(\mathcal{O}_{\mathbb{C}P^p}(-1)^p) = (c_1(\mathcal{O}_{\mathbb{C}P^p}(-1))^p = 1$. ■

Using Lemma 4.14, one can conclude the proof of Theorem 1.6. ■

Remark 4.15. As in Proposition 2.1 point (b), we also need to consider towers of fibrations that are finite covers of the original ones. Note that any covering of $\mathbb{Q}^{(k),(p),g}_\lambda$ must also have nontrivial PGW cf. Proposition 3.7(iii). □

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Notes

[1]To simplify the notation throughout the paper, we will use the same symbol for a map, for example F, and its homotopy class $[F]$ whenever the distinction is not really relevant.

[2]Recall that in a topological group G_λ, the Samelson product between two elements $\alpha \in \pi_p G_\lambda$ and $\beta \in \pi_q G_\lambda$ is an element in $\pi_{p+q} G_\lambda$ given by the homotopy class of the anticommutator map $[\alpha, \beta] : S^p \times S^q / S^p \vee S^q \longrightarrow G_\lambda$ given by $[\alpha, \beta](u, v) = \alpha(u)\beta(v)\alpha^{-1}(u)\beta^{-1}(v)$. As explained in Whitehead [Chapter X] [23] this is, up to a sign, equal to the desuspension of the Whitehead product between the suspension of α and β in the classifying space $\mathrm{B}G_\lambda$.

[3]The topologies of spaces $\mathcal{F}^l_{g,m}(\Omega, s_D)\overline{\mathcal{F}}^l_{g,m}(\Omega, s_D)$ and $\mathrm{Hom}(T\mathrm{Reg}(\Sigma), f^*T\Omega_b^{\mathrm{vert}})$ can be introduced exactly as in Li–Tian [15].

References

[1] Abreu, M. "Topology of symplectomorphism groups of $S^2 \times S^2$." *Inventiones Mathematicae* 131 (1998): 1–23.

[2] Abreu, M., G. Granja, and N. Kitchloo. "Compatible complex structures on symplectic rational ruled surfaces." *Duke Mathematical Journal* (forthcoming): SG/0610436.

[3] Abreu, M., and D. McDuff. "Topology of symplectomorphism groups of rational ruled surfaces." *Journal of the American Mathematical Society* 13, no. 4 (2000): 971–1009.

[4] Anjos, S.. "Homotopy of symplectomorphism groups of $S^2 \times S^2$." *Geometry & Topology* 6 (2002): 195–218.

[5] Anjos, S., and G. Granja. "Homotopy decomposition of a group of symplectomorphisms of $S^2 \times S^2$." *Topology* 43 (2004): 599–618.

[6] Andrews, P., and M. Arkovitz. "Sullivan's minimal model and higher order Whitehead products." *Canadian Journal of Mathematics* 13 (1978): 961–82.

[7] Allday, C. "Rational Whitehead products and a spectral sequence of Quillen, II." *Houston Journal of Mathematics* 3, no. 3 (1977): 301–8.

[8] Allday, C. "Rational Whitehead products and a spectral sequence of Quillen." *Pacific Journal of Mathematics* 46, no. 2 (1973): 313–23.

[9] Audin, M. "Hamiltonians on Four-dimensional Compact Symplectic Manifolds." In *Geometrie Symplectique et mecanique*. Lecture Notes in Mathematics 1416. Berlin: Springer, 1996.

[10] Buşe, O. "Relative family Gromov–Witten invariants and symplectomorphisms." *Pacific Journal of Mathematics* 218, no. 2 (2005): 315–41.

[11] Gromov, M. "Pseudo holomorphic curves in symplectic manifolds." *Inventiones Mathematicae* 82 (1985): 307–47.

[12] Guillemin, V., E. Lerman, and S. Sternberg. *Symplectic Fibrations and Multiplicity Diagrams*. Cambridge: Cambridge University Press, 1996.

[13] Karshon, Y. "Periodic Hamiltonian flows on four-dimensional manifolds." *Memoirs of the American Mathematical Society* 141, no. 672 (1999): viii+71.

[14] Le, H. V., and K. Ono. "Parametrized Gromov–Witten and topology of symplectomorphism groups." *Advanced Studies in Pure Mathematics* 52, (2008): 51–75.

[15] Li, J., and G. Tian. "Virtual Moduli Cycles and Gromov–Witten Invariants of General Symplectic Manifolds." In *Topics in 4-manifolds, (Irvine, CA, 1996)*, 47–83. Boston: First International Press Lecture Series I. International Press, 1996.

[16] McDuff, D. "Symplectomorphism Groups and Almost Complex Structures." In *Essays on Geometry and Related Topics, Vols. 1 and 2*, 527–56. Enseignement Mathématique. Geneva: 2001.

[17] McDuff, D., and D. A. Salamon. *J-holomorphic Curves and Quantum Cohomology*. University Lecture Series 6. Providence, RI: American Mathematical Society, 1994.

[18] D. McDuff and D. A. Salamon, *Introduction to Symplectic Topology*, 2nd ed. Oxford: Oxford University Press, 1998.

[19] Porter, G. "Higher order Whitehead products." *Topology* 3 (1965): 123–35.

[20] Ruan, Y. "Virtual Neighborhoods and Pseudo-holomorphic Curves." In *Proceedings of the 6th Gokova Conference*, 161–231. Turkish Journal of Mathematics 23, no. 1, 1999.

[21] Seidel, P. "π_1 of symplectic automorphism groups and invertibles in quantum homology rings." *Geometric and Functional Analysis* 7, no. 6 (1997): 1046–95.

[22] Stasheff, J. "H-spaces from a homotopy point of view." *Lecture Notes in Mathematics* 161 (1970): v+95.

[23] Whitehead, G. H. *Elements of Homotopy Theory*. Graduate Texts in Mathematics 61. New York: Springer, 1978.