

Large n Limit of Gaussian Random Matrices with External Source, Part III: Double Scaling Limit

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Received: 28 February 2006 / Accepted: 5 July 2006
Published online: 9 December 2006 – © Springer-Verlag 2006

Abstract: We consider the double scaling limit in the random matrix ensemble with an external source

$$\frac{1}{Z_n} e^{-n\text{Tr}(\frac{1}{2}M^2 - AM)} dM$$

defined on $n \times n$ Hermitian matrices, where A is a diagonal matrix with two eigenvalues $\pm a$ of equal multiplicities. The value $a = 1$ is critical since the eigenvalues of M accumulate as $n \rightarrow \infty$ on two intervals for $a > 1$ and on one interval for $0 < a < 1$. These two cases were treated in Parts I and II, where we showed that the local eigenvalue correlations have the universal limiting behavior known from unitary random matrix ensembles. For the critical case $a = 1$ new limiting behavior occurs which is described in terms of Pearcey integrals, as shown by Brézin and Hikami, and Tracy and Widom. We establish this result by applying the Deift/Zhou steepest descent method to a 3×3 -matrix valued Riemann-Hilbert problem which involves the construction of a local parametrix out of Pearcey integrals. We resolve the main technical issue of matching the local Pearcey parametrix with a global outside parametrix by modifying an underlying Riemann surface.

1. Introduction and Statement of Results

1.1. The random matrix model. This is the third and final part of a sequence of papers on the Gaussian random matrix ensemble with external source

$$\frac{1}{Z_n} e^{-n\text{Tr}(\frac{1}{2}M^2 - AM)} dM, \quad (1.1)$$

* The first author was supported in part by the National Science Foundation (NSF) Grant DMS-0354962.

** The second author was supported by FWO-Flanders project G.0455.04, by K.U. Leuven research grant OT/04/24, by INTAS Research Network 03-51-6637, by a grant from the Ministry of Education and Science of Spain, project code MTM2005-08648-C02-01, and by the European Science Foundation Program MISGAM.

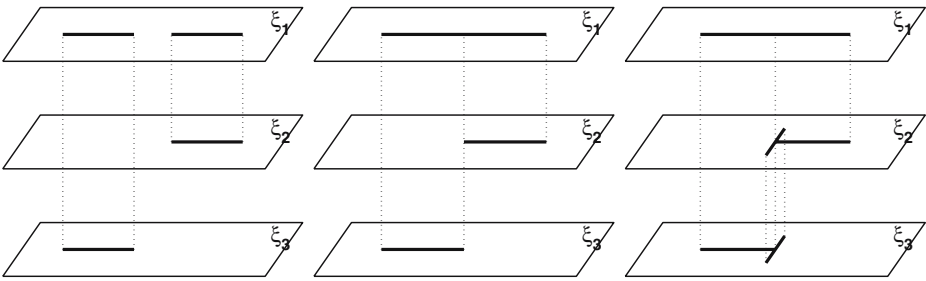


Fig. 1. The structure of the Riemann surface for Eq. (1.2) for the values $a > 1$ (left), $a = 1$ (middle) and $a < 1$ (right). In all cases the eigenvalues of M accumulate on the interval(s) of the first sheet with a density given by (1.3)

defined on $n \times n$ Hermitian matrices, where A is a diagonal matrix with two eigenvalues $\pm a$ (with $a > 0$) of equal multiplicities (so that, n is even). This matrix ensemble was introduced by Brézin and Hikami [9, 10] as a simple model for a phase transition that is expected to exhibit universality properties. The phase transition can be seen from the behavior of the eigenvalues of M in the large n limit, since for $a > 1$, the eigenvalues accumulate on two intervals, while for $0 < a < 1$, the eigenvalues accumulate on one interval. The limiting mean density of eigenvalues follows from earlier work of Pastur [22]. It is based on an analysis of the equation (Pastur equation)

$$\xi^3 - z\xi^2 + (1 - a^2)\xi + a^2z = 0, \tag{1.2}$$

which yields an algebraic function $\xi(z)$ defined on a three-sheeted Riemann surface. The restrictions of $\xi(z)$ to the three sheets are denoted by $\xi_j(z)$, $j = 1, 2, 3$. There are four real branch points if $a > 1$ which determine two real intervals. The two intervals come together for $a = 1$, and for $0 < a < 1$, there are two real branch points, and two purely imaginary branch points. Figure 1 depicts the structure of the Riemann surface $\xi(z)$ for $a > 1$, $a = 1$, and $a < 1$.

In all cases we have that the limiting mean eigenvalue density $\rho(x) = \rho(x; a)$ of the matrix M from (1.1) is given by

$$\rho(x; a) = \frac{1}{\pi} \text{Im } \xi_{1+}(x), \quad x \in \mathbb{R}, \tag{1.3}$$

where $\xi_{1+}(x)$ denotes the limiting value of $\xi_1(z)$ as $z \rightarrow x$ with $\text{Im } z > 0$. For $a = 1$ the limiting mean eigenvalue density vanishes at $x = 0$ and $\rho(x; a) \sim |x|^{1/3}$ as $x \rightarrow 0$.

We note that this behavior at the closing (or opening) of a gap is markedly different from the behavior that occurs in the usual unitary random matrix ensembles $Z_n^{-1} e^{-n\text{Tr}V(M)} dM$, where a closing of the gap in the spectrum typically leads to a limiting mean eigenvalue density ρ that satisfies $\rho(x) \sim (x - x^*)^2$ as $x \rightarrow x^*$ if the gap closes at $x = x^*$. In that case the local eigenvalue correlations can be described in terms of ψ -functions associated with the Painlevé II equation, see [5, 11]. The phase transition for the model (1.1) is different, and it cannot be realized in a unitary random matrix ensemble.

1.2. Non-intersecting Brownian motion. The nature of the phase transition at $a = 1$ may also be seen from an equivalent model of non-intersecting Brownian paths, see

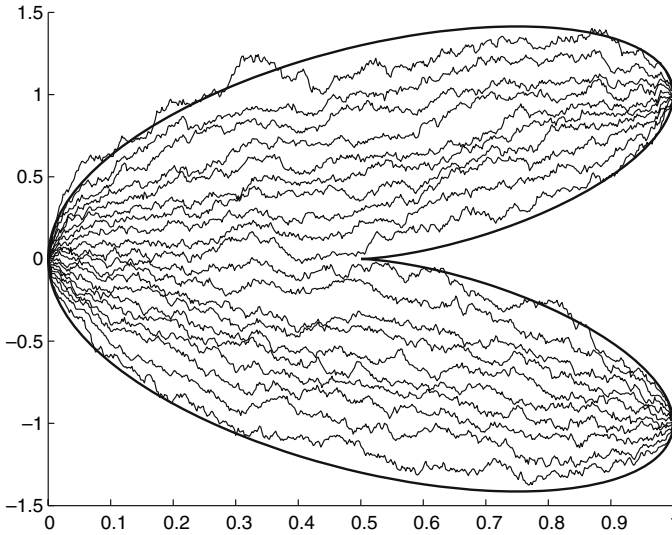


Fig. 2. Non-intersecting Brownian paths that start at one point and end at two points. At any intermediate time the positions of the paths are distributed as the eigenvalues of a Gaussian random matrix ensemble with external source. As their number increases the paths fill out a region whose boundary has a cusp

Fig. 2. Consider n independent one-dimensional Brownian motions that are conditioned to start at time $t = 0$ at the origin, end at time $t = 1$ at ± 1 , where half of the paths ends at $+1$ and the other half at -1 , and that are conditioned not to intersect at intermediate times $t \in (0, 1)$. As explained in [2], at any intermediate time t , the positions of the n Brownian motions, have the same distribution as the eigenvalues of a Gaussian random matrix ensemble with external source (up to trivial scaling). Now, as $n \rightarrow \infty$ and under appropriate scaling of the variance of the Brownian motions, the paths fill out a region in the $t - x$ -plane. Then for small time the paths are in one group, which at a certain critical time t_{cr} splits into two groups, where one group ends at $x = +1$ and the other group at $x = -1$. The situations $t < t_{cr}$, $t = t_{cr}$, and $t > t_{cr}$ correspond to $a < 1$, $a = 1$, and $a > 1$, respectively, in the Gaussian random matrix model with external source.

The boundary curve has a cusp singularity at the critical time as shown in Fig. 2.

1.3. Correlation kernel. Brézin and Hikami [9, 10] showed, see also [28], that the eigenvalues of the random matrix ensemble (1.1) are distributed according to a determinantal point process. There is a kernel $K_n(x, y; a)$ so that the eigenvalues x_1, \dots, x_n have the joint probability density

$$p_n(x_1, \dots, x_n) = \frac{1}{n!} \det(K_n(x_j, x_k; a))_{j,k=1,\dots,n}$$

and so that for each $m \leq n$, the m -point correlation function

$$R_m(x_1, \dots, x_m) = \frac{n!}{(n - m)!} \underbrace{\int \cdots \int}_{n-m \text{ times}} p_n(x_1, \dots, x_n) dx_{m+1} \cdots dx_n$$

takes determinantal form as well:

$$R_m(x_1, \dots, x_m) = \det(K_n(x_j, x_k; a))_{j,k=1,\dots,m}.$$

In [6] we pointed out that the kernel can be built out of multiple Hermite polynomials [3, 8, 26] in much the same way that the correlation kernel for unitary random matrix ensembles (without external source) is related to orthogonal polynomials. The Christoffel-Darboux formula for multiple orthogonal polynomials [6, 14] allows one to express the kernel in terms of the Riemann-Hilbert problem for multiple Hermite polynomials (see [27] and below). Applying the Deift/Zhou steepest descent analysis [15, 18] to the Riemann-Hilbert problem in the non-critical case, we were able to show that the kernel has the usual scaling limits from random matrix theory. That is, we obtain the sine kernel

$$K^{bulk}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)} \tag{1.4}$$

in the bulk, and the Airy kernel

$$K^{edge}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} \tag{1.5}$$

at the edge of the spectrum, as scaling limits of $K_n(x, y; a)$ if $a > 1$ [7] or $0 < a < 1$ [2].

1.4. Double scaling limit. In this paper we consider the double scaling limit at the critical parameter $a = 1$ of the Gaussian random matrix ensemble with external source, or equivalently, of the non-intersecting Brownian motion model at the critical time $t = t_{cr}$. As is usual in a critical case, there is a family of limiting kernels that arise when a changes with n and $a \rightarrow 1$ as $n \rightarrow \infty$ in a critical way. These kernels are constructed out of Pearcey integrals and therefore they are called Pearcey kernels. The Pearcey kernels were first described by Brézin and Hikami [9, 10]. A detailed proof of the following result was recently given by Tracy and Widom [25].

Theorem 1.1. *We have for every fixed $b \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{3/4}} K_n \left(\frac{x}{n^{3/4}}, \frac{y}{n^{3/4}}; 1 + \frac{b}{2\sqrt{n}} \right) = K^{cusp}(x, y; b), \tag{1.6}$$

where K^{cusp} is the Pearcey kernel

$$K^{cusp}(x, y; b) = \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - bp(x)q(y)}{x - y} \tag{1.7}$$

with

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2 + isx} ds \quad \text{and} \quad q(y) = \frac{1}{2\pi} \int_{\Sigma} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 + ity} dt. \tag{1.8}$$

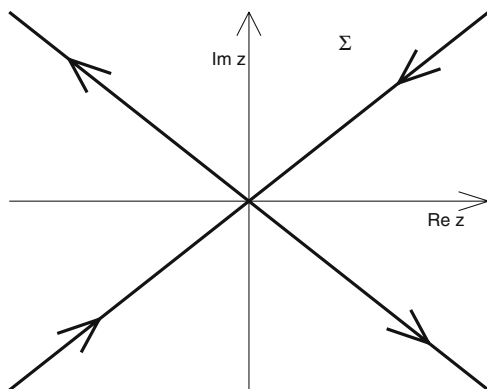


Fig. 3. The contour Σ that appears in the definition of $q(y)$

The contour Σ consists of the four rays $\arg z = \pm\pi/4, \pm 3\pi/4$, with the orientation shown in Fig. 3.

The functions (1.8) are called Pearcey integrals [23]. They are solutions of the third order differential equations $p'''(x) = xp(x) + bp'(x)$ and $q'''(y) = -yq(y) + bq'(y)$, respectively. Away from the critical point $x = 0$, the usual scaling limits (1.4) and (1.5) from random matrix theory continue to hold in the case $a = 1$ (also in the double scaling regime). This can be proved for example as in [7, 25], and we will not consider this any further here.

Theorem 1.1 implies that local eigenvalue statistics of eigenvalues near 0 are expressed in terms of the Pearcey kernel. For example we have the following corollary of Theorem 1.1.

Corollary 1.2. *The probability that a matrix of the ensemble (1.1) with $a = 1 + bn^{-1/2}/2$ has no eigenvalues in the interval $[cn^{-3/4}, dn^{-3/4}]$ converges, as $n \rightarrow \infty$, to the Fredholm determinant of the integral operator with kernel $K^{cusp}(x, y; b)$ acting on $L^2(c, d)$.*

Similar expressions hold for the probability to have one, two, three, ..., eigenvalues in an $O(n^{-3/4})$ neighborhood of $x = 0$.

Tracy and Widom [25] and Adler and van Moerbeke [1] gave differential equations for the gap probabilities associated with the Pearcey kernel and with the more general Pearcey process which arises from considering the non-intersecting Brownian motion model at several times near the critical time. See also [20] where the Pearcey process appears in a combinatorial model on random partitions.

1.5. Steepest descent method for RH problems. Brézin and Hikami and also Tracy and Widom used a double integral representation for the kernel in order to establish Theorem 1.1. In this paper we use the Deift/Zhou steepest descent method for the Riemann-Hilbert problem for multiple Hermite polynomials. This method is less direct than the steepest descent method for integrals. However, an approach based on the Riemann-Hilbert problem may be applicable to more general situations, where an integral representation is not available. This is the case, for example, for the general

(non-Gaussian) unitary random matrix ensemble with external source

$$\frac{1}{Z_n} e^{-n\text{Tr}(V(M)-AM)} dM \tag{1.9}$$

with a general potential V . The Riemann-Hilbert problem is formulated in Sect. 2.

The asymptotic analysis of the Riemann-Hilbert problem presents a new feature that we feel is of importance in its own right. We will not use the Pastur equation (1.2) which defines the ξ -functions and the Riemann surface that corresponds to it, but instead we use a modified equation to define the ξ -functions. We discuss this in Sect. 3. The modification may be thought of in potential theoretic terms and we briefly discuss this in Sect. 3 as well.

The anti-derivatives of the modified ξ -functions are introduced in Sect. 4 and they play an important role in the steepest descent analysis of the Riemann-Hilbert problem in the rest of the paper. The main issue is the construction in Sect. 8 of the local parametrix around 0 with the aid of Pearcey integrals. The modification of the ξ -functions is used here to be able to match the local Pearcey parametrix with the outside parametrix. Even so it turns out that we cannot achieve the matching condition on a fixed circle around the origin, but only on circles with radii $n^{-1/4}$ that decrease as n increases. However, the circles are big enough to capture the behavior (1.6) which takes place at a distance to the origin of order $n^{-3/4}$. The precise estimates that lead to the proof of Theorem 1.1 are given in the final Sects. 9 and 10.

2. Riemann-Hilbert Problem

As shown in our paper [6], the correlation kernel is expressed in terms of the solution to the following 3×3 matrix valued Riemann-Hilbert (RH) problem.

Find $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{3 \times 3}$ such that

- Y is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- for $x \in \mathbb{R}$, we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n(\frac{1}{2}x^2-ax)} & e^{-n(\frac{1}{2}x^2+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $Y_+(x)$ ($Y_-(x)$) denotes the limit of $Y(z)$ as $z \rightarrow x$ from the upper (lower) half-plane,

- as $z \rightarrow \infty$, we have

$$Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}, \tag{2.2}$$

where I denotes the 3×3 identity matrix.

The RH problem has a unique solution, given explicitly in terms of the multiple Hermite polynomials. The correlation kernel of the Gaussian random matrix model with external source is equal to

$$K_n(x, y; a) = \frac{e^{-\frac{1}{4}n(x^2+y^2)}}{2\pi i(x-y)} (0 \ e^{nay} \ e^{-nay}) Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.3}$$

In what follows we are going to apply the Deift/Zhou steepest descent method for RH problems to the above RH problem for Y . It consists of a sequence of explicit transformations $Y \mapsto T \mapsto S \mapsto R$ which leads to a RH problem for R in which all jumps are close to the identity matrix and which is normalized at infinity. Then R is close to the identity matrix, and analyzing the effect of the transformations on the kernel (2.3) we will be able to prove Theorem 1.1.

3. Modification of the ξ -Functions

3.1. Modified Pastur equation. The analysis in [2, 7] for the cases $a > 1$ and $0 < a < 1$ was based on Eq. (1.2) and it would be natural to use (1.2) also in the case $a = 1$. Indeed, that is what we tried to do, and we found that it works for $a \equiv 1$, but in the double scaling regime $a = 1 + \frac{b}{2\sqrt{n}}$ with $b \neq 0$, it led to problems that we were unable to resolve in a satisfactory way. A crucial feature of our present approach is a modification of Eq. (1.2) when a is close to 1, but different from 1. At $x = 0$ we wish to have a double branch point for all values of a so that the structure of the Riemann surface is as in the middle figure of Fig. 1 for all a .

For $c > 0$, we consider the Riemann surface for the equation

$$z = \frac{w^3}{w^2 - c^2}, \tag{3.1}$$

where w is a new auxiliary variable. The Riemann surface has branch points at $z^* = \frac{3\sqrt{3}}{2}c$, $-z^*$ and a double branch point at 0. There are three inverse functions w_k , $k = 1, 2, 3$, that behave as $z \rightarrow \infty$ as

$$\begin{aligned} w_1(z) &= z - \frac{c^2}{z} + O\left(\frac{1}{z^3}\right), \\ w_2(z) &= c + \frac{c^2}{2z} + O\left(\frac{1}{z^2}\right), \\ w_3(z) &= -c + \frac{c^2}{2z} + O\left(\frac{1}{z^2}\right), \end{aligned} \tag{3.2}$$

and which are defined and analytic on $\mathbb{C} \setminus [-z^*, z^*]$, $\mathbb{C} \setminus [0, z^*]$ and $\mathbb{C} \setminus [-z^*, 0]$, respectively.

Then we define the modified ξ -functions

$$\xi_k = w_k + \frac{p}{w_k}, \quad \text{for } k = 1, 2, 3, \tag{3.3}$$

which we also consider on their respective Riemann sheets. In what follows we take

$$c = \frac{a + \sqrt{a^2 + 8}}{4} \quad \text{and} \quad p = c^2 - 1. \tag{3.4}$$

Note that $a = 1$ corresponds to $c = 1$ and $p = 0$. In that case the functions coincide with the solutions of Eq. (1.2) that we used in our earlier works. From (3.1), (3.3), and (3.4) we obtain the modified Pastur equation

$$\xi^3 - z\xi^2 + (1 - a^2)\xi + a^2z + \frac{(c^2 - 1)^3}{c^2z} = 0, \tag{3.5}$$

where c is given by (3.4).

Lemma 3.1. *Let $a > 0$ and take c and p as in (3.4). Then at infinity we have*

$$\begin{aligned} \xi_1(z) &= z - \frac{1}{z} + O\left(\frac{1}{z^3}\right), \\ \xi_2(z) &= a + \frac{1}{2z} + O\left(\frac{1}{z^2}\right), \\ \xi_3(z) &= -a + \frac{1}{2z} + O\left(\frac{1}{z^2}\right). \end{aligned} \tag{3.6}$$

Proof. This follows from direct calculations using (3.2), (3.3) and the fact that $2c - \frac{1}{c} = a$. Alternatively, one could also use (3.5). \square

The new ξ -functions have the same asymptotic behavior (3.6) as $z \rightarrow \infty$ (up to order $1/z^2$) as the solutions of (1.2). This is important for the first two transformations of the Riemann-Hilbert problem. The situation at $z = 0$ is different. The fact that we can control the behavior at $z = 0$ as well is the reason for the introduction of the modified ξ -functions.

3.2. *Behavior at $z = 0$.* We start with the behavior of the functions w_k .

Lemma 3.2. *There exist analytic functions f_1 and g_1 defined in a neighborhood U_1 of $z = 0$ so that for $z \in U_1$ and $k = 1, 2, 3$,*

$$w_k(z) = \begin{cases} -\omega^{2k} z^{1/3} f_1(z) - \omega^k z^{5/3} g_1(z) + \frac{z}{3} & \text{for } \text{Im } z > 0, \\ -\omega^k z^{1/3} f_1(z) - \omega^{2k} z^{5/3} g_1(z) + \frac{z}{3} & \text{for } \text{Im } z < 0. \end{cases} \tag{3.7}$$

In addition, we have $f_1(0) = c^{2/3}$, and $f_1(z)$ and $g_1(z)$ are real for real $z \in U_1$.

Proof. Putting $z = x^3$ and $w = xy$ in (3.1) we obtain

$$y^3 = x^2 y^2 - c^2, \tag{3.8}$$

which has a solution $y = y(x)$ that is analytic in a neighborhood U_1 of 0 and satisfies $y(0) = -c^{2/3}$ and $y'(0) = 0$. Then we can write $y(x) = -f_1(x^3) - x^4 g_1(x^3) + x^2 h_1(x^3)$ with f_1, g_1 and h_1 analytic in U_1 and $f_1(0) = c^{2/3}$. Putting this back in (3.8) we find after straightforward calculations that (with $z = x^3$)

$$f_1(z)g_1(z) = \frac{1}{9}, \quad f_1(z)^3 - c^2 + \frac{2}{27}z^2 + g_1(z)^3 z^4 = 0, \tag{3.9}$$

and $h_1(z) = \frac{1}{3}$. Going back to z and w variables, we see that there is a solution $w = w(z)$ to (3.1) with

$$w(z) = -z^{1/3} f_1(z) - z^{5/3} g_1(z) + \frac{z}{3}, \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0],$$

where we take the principal branches of the fractional powers. This solution is real for z real and positive, and so it coincides with the solution $w_3(z)$. This proves (3.7) for $k = 3$. The expressions (3.7) for $k = 1, 2$ follow by analytic continuation.

Since $y(x)$ is real for real x , we also find that $f_1(z)$ and $g_1(z)$ are real if z is real. \square

From (3.9) it is easy to give explicit expressions for f_1 and g_1 . However we will not use this in the future.

From Lemma 3.2 and (3.3) we get the following behavior for the functions ξ_k near $z = 0$.

Lemma 3.3. *There exist analytic functions f_2 and g_2 defined in a neighborhood U_2 of $z = 0$ so that for $z \in U_2$ and $k = 1, 2, 3$,*

$$\xi_k(z) = \begin{cases} -\omega^{2k} z^{1/3} f_2(z) - \omega^k z^{-1/3} g_2(z) + \frac{z}{3} & \text{for } \text{Im } z > 0, \\ -\omega^k z^{1/3} f_2(z) - \omega^{2k} z^{-1/3} g_2(z) + \frac{z}{3} & \text{for } \text{Im } z < 0. \end{cases} \tag{3.10}$$

In addition, we have

$$f_2(0) = c^{2/3} + \frac{1}{3}c^{-4/3}(c^2 - 1), \quad g_2(0) = c^{-2/3}(c^2 - 1), \tag{3.11}$$

and $f_2(z)$ and $g_2(z)$ are real for real $z \in U_2$.

Proof. This follows from (3.3) and the previous lemma. Indeed from (3.7) where f_1 and g_1 satisfy the relations (3.9), we can deduce

$$\frac{1}{w_3(z)} = -z^{1/3} \frac{1}{3c^2}(f_1(z) + 3z^2 g_1(z)^2) - z^{-1/3} \frac{1}{3c^2}(3f_1(z)^2 + z^2 g_1(z)).$$

Then (3.10) follows from (3.7) and (3.3) if we take

$$f_2(z) = f_1(z) + \frac{c^2 - 1}{3c^2}(f_1(z) + 3z^2 g_1(z)^2) \tag{3.12}$$

and

$$g_2(z) = z^2 g_1(z) + \frac{c^2 - 1}{3c^2}(3f_1(z)^2 + z^2 g_1(z)). \tag{3.13}$$

Because of Lemma 3.2 this also implies (3.11) and the fact that $f_2(z)$ and $g_2(z)$ are real for real $z \in U_2$. \square

3.3. Potential theoretic interpretation. As an aside we want to mention that the modified ξ -functions may be thought of in terms of a modified equilibrium problem for logarithmic potentials. For $a > 1$, it was noted in [7], that the limiting mean eigenvalue density $\rho(x) = \rho(x; a)$ may be characterized as follows. We minimize

$$\begin{aligned} E(\mu_1, \mu_2) &= \iint \log \frac{1}{|x - y|} d\mu_1(x) d\mu_1(y) + \iint \log \frac{1}{|x - y|} d\mu_2(x) d\mu_2(y) \\ &+ \iint \log \frac{1}{|x - y|} d\mu_1(x) d\mu_2(y) + \int \left(\frac{1}{2}x^2 - ax \right) d\mu_1(x) \\ &+ \int \left(\frac{1}{2}x^2 + ax \right) d\mu_2(x) \end{aligned} \tag{3.14}$$

among all non-negative measures μ_1, μ_2 on \mathbb{R} with $\int d\mu_1 = \int d\mu_2 = \frac{1}{2}$. There is a unique minimizer [24], and for $a > 1$, we have that $\text{supp}(\mu_1) \subset [0, \infty)$, $\text{supp}(\mu_2) \subset$

$(-\infty, 0]$, and ρ is the density of $\mu_1 + \mu_2$. For $a < 1$, the minimizing measures for (3.14) do not have disjoint supports, and in fact these minimizers are not related to our random matrix ensemble (1.1) at all.

The modification we are alluding to is to minimize (3.14) among *signed* measures $\mu_1 = \mu_1^+ - \mu_1^-$, $\mu_2 = \mu_2^+ - \mu_2^-$, where μ_j^\pm are non-negative measures, that satisfy $\int d\mu_1 = \int d\mu_2 = \frac{1}{2}$ and in addition

- (1) $\text{supp}(\mu_1) \subset [0, \infty)$, $\text{supp}(\mu_2) \subset (-\infty, 0]$, and,
- (2) there is a $\delta > 0$, such that $\text{supp}(\mu_1^-) \subset [0, \delta)$ and $\text{supp}(\mu_2^-) \subset (-\delta, 0]$.

The condition (1) plays a role for $a < 1$, since it prevents the supports of μ_1 and μ_2 to overlap. For $a > 1$, condition (2) plays a role, since it allows the measures to become negative near 0. Now let μ_1, μ_2 be the minimizers for this modified equilibrium problem, and let $\tilde{\rho}$ be the density of $\mu_1 + \mu_2$. Then it can be shown that the density of $\mu_1 + \mu_2$ is equal to $\frac{1}{\pi} \text{Im} \xi_{1+}(x)$, where ξ_1 is the modified ξ_1 -function introduced in this section.

We will not use this potential-theoretic connection in the analysis that follows in this paper, but we anticipate that it might be important for the general unitary random matrix ensemble with external source (1.9).

We finally note that a modified equilibrium problem was also used in [11, 12] in order to analyse the double scaling limit in unitary random matrix ensembles (without external source), so one might speculate that such an approach might be characteristic for double scaling limits in random matrix ensembles.

4. The λ -Functions

4.1. Definition and first properties. The main role is played by the λ -functions which are anti-derivatives of the ξ -functions. They are defined here as

$$\lambda_k(z) = \int_{0+}^z \xi_k(s) ds, \tag{4.1}$$

where the path of integration starts at 0 on the upper side of the cut and is fully contained (except for the initial point) in $\mathbb{C} \setminus (-\infty, z^*]$ for $k = 1, 2$, and in $\mathbb{C} \setminus (-\infty, 0]$ for $k = 3$. Then λ_1 and λ_2 are defined and analytic on $\mathbb{C} \setminus (-\infty, z^*]$, and λ_3 is defined and analytic on $\mathbb{C} \setminus (-\infty, 0]$.

As follows from (3.6) and (4.1), the λ -functions behave at infinity as

$$\begin{aligned} \lambda_1(z) &= \frac{1}{2}z^2 - \log z + \ell_1 + O(1/z), \\ \lambda_2(z) &= az + \frac{1}{2} \log z + \ell_2 + O(1/z), \\ \lambda_3(z) &= -az + \frac{1}{2} \log z + \ell_3 + O(1/z), \end{aligned} \tag{4.2}$$

for certain constants $\ell_k, k = 1, 2, 3$, where $\log z$ is taken as the principal value, that is, with a cut along the negative real axis.

From contour integration based on (3.6) where we use the residue of ξ_2 at infinity, we find $\lambda_{1-}(0) = \pi i$ and $\lambda_{2-}(0) = -\pi i$. Then we get the following jump properties of the λ -functions on the cuts $(-\infty, 0]$ and $(-\infty, z^*]$:

$$\begin{aligned} \lambda_{1+} &= \lambda_{2-} + \pi i, & \lambda_{2+} &= \lambda_{1-} - \pi i, & \lambda_{3+} &= \lambda_{3-}, & \text{on } [0, z^*], \\ \lambda_{1+} &= \lambda_{3-}, & \lambda_{2+} &= \lambda_{2-} + \pi i, & \lambda_{3+} &= \lambda_{1-} - \pi i, & \text{on } [-z^*, 0], \\ \lambda_{1+} &= \lambda_{1-} - 2\pi i, & \lambda_{2+} &= \lambda_{2-} + \pi i, & \lambda_{3+} &= \lambda_{3-} + \pi i, & \text{on } (-\infty, -z^*]. \end{aligned} \tag{4.3}$$

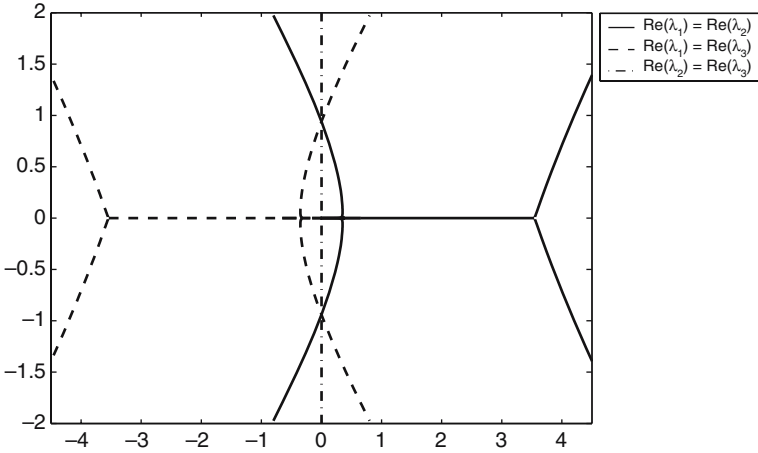


Fig. 4. Curves where $\text{Re } \lambda_j = \text{Re } \lambda_k$ for the value $a = 2.0$

4.2. Behavior near $z = 0$. Near the origin the λ -functions behave as follows.

Lemma 4.1. *There exist analytic functions f_3 and g_3 in a neighborhood U_3 of $z = 0$ so that*

$$\lambda_k(z) = \begin{cases} -\frac{3}{4}\omega^{2k}z^{4/3}f_3(z) - \frac{1}{2}\omega^kz^{2/3}g_3(z) + \frac{z^2}{6} & \text{for } \text{Im } z > 0, \\ \lambda_{k-}(0) - \frac{3}{4}\omega^kz^{4/3}f_3(z) - \frac{1}{2}\omega^{2k}z^{2/3}g_3(z) + \frac{z^2}{6} & \text{for } \text{Im } z < 0. \end{cases} \quad (4.4)$$

In addition, we have

$$f_3(0) = f_2(0) = c^{2/3} + \frac{1}{3}c^{-4/3}(c^2 - 1), \quad g_3(0) = 3g_2(0) = 3c^{-2/3}(c^2 - 1), \quad (4.5)$$

and $f_3(z)$ and $g_3(z)$ are real for real $z \in U_3$.

Proof. The relations (4.4) follow by integrating (3.10). Note that $\lambda_{1-}(0) = \pi i$, $\lambda_{2-}(0) = -\pi i$, and $\lambda_{3-}(0) = 0$. The other statements of the lemma also follow directly from Lemma 3.3. \square

4.3. Critical trajectories. Curves where $\text{Re } \lambda_j = \text{Re } \lambda_k$ for some $j \neq k$ are shown in Figs. 4, 5, and 6, for the cases $a > 1$, $a = 1$, and $a < 1$, respectively. These are critical curves that play a crucial role in the asymptotic analysis. The curves are critical trajectories of the quadratic differentials $(\xi_j(z) - \xi_k(z))^2 dz^2$ (and their analytic continuations beyond the branch cuts in case $a < 1$).

The solid curves in Figs. 4–6 are the critical trajectories of the quadratic differential $(\xi_1(z) - \xi_2(z))^2 dz^2$. The quadratic differential has a simple zero at $z = z^*$. Three trajectories are emanating from $z = z^*$ at equal angles, one of these being the real interval $(0, z^*)$. For $a > 1$, the quadratic differential has a double zero at $z = x_0$ for some $x_0 \in (0, z^*)$. Four trajectories are emanating from the double zero at equal angles as can be seen in Fig. 4.

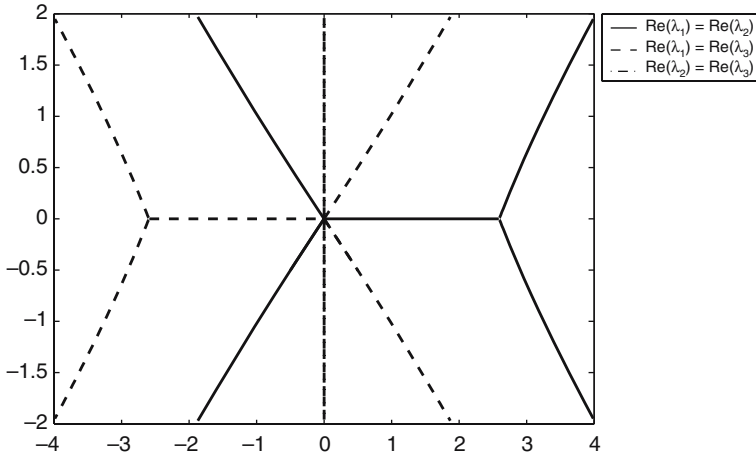


Fig. 5. Curves where $\text{Re } \lambda_j = \text{Re } \lambda_k$ for the value $a = 1.0$

The dashed curves are the critical trajectories of the quadratic differential $(\xi_3(z) - \xi_3(z))^2 dz^2$. Because of symmetry, these are the mirror images of the trajectories of the quadratic differential $(\xi_1(z) - \xi_2(z))^2 dz^2$ with respect to the imaginary axis. For $a > 1$, the solid curve that passes vertically through x_0 and its dashed mirror image with respect to the imaginary axis meet in two points $\pm iy_0$ on the imaginary axis. Together they enclose a neighborhood of the origin.

The dashed-dotted curves are the critical trajectories of $(\xi_2(z) - \xi_3(z))^2 dz^2$. For $a < 1$, the quadratic differential has two double zeros at $z = \pm iy_0$ for some $y_0 > 0$. Four trajectories are emanating from these double zeros at equal angles as shown in Fig. 6. Besides the imaginary axis there are curves passing horizontally through $\pm iy_0$, and these curves meet each other at two points $\pm x_0$ on the real axis and they enclose a neighborhood of the origin. Beyond these two points the quadratic differentials have analytic continuations, but the formula changes since either ξ_2 or ξ_3 reaches its branch cut and changes into ξ_1 . Consequently, the dashed-dotted curves in Fig. 6 continue beyond $\pm x_0$ as either solid or dashed curves.

The relative orderings of the real parts $\text{Re } \lambda_1$, $\text{Re } \lambda_2$ and $\text{Re } \lambda_3$ changes if we cross one of the critical trajectories, but it remains constant in the regions bounded by the critical trajectories. For each of the unbounded regions we can determine the ordering from the behavior at infinity (4.2). For example, we have in the right-most region $\text{Re } \lambda_1 > \text{Re } \lambda_2 > \text{Re } \lambda_3$, and if we cross the solid curve where $\text{Re } \lambda_1 = \text{Re } \lambda_2$, the ordering becomes $\text{Re } \lambda_2 > \text{Re } \lambda_1 > \text{Re } \lambda_3$, and so on.

In the cases $a < 1$ and $a > 1$ the trajectories enclose a bounded neighborhood of the origin. There is no such neighborhood in case $a = 1$. The neighborhood is small if a is close to 1. In this neighborhood the relative ordering of the real parts is different.

So we can easily verify the following.

Lemma 4.2. *Except for z in the exceptional bounded neighborhood of the origin, we have that*

$$\text{Re } \lambda_2(z) > \max(\text{Re } \lambda_1(z), \text{Re } \lambda_3(z))$$

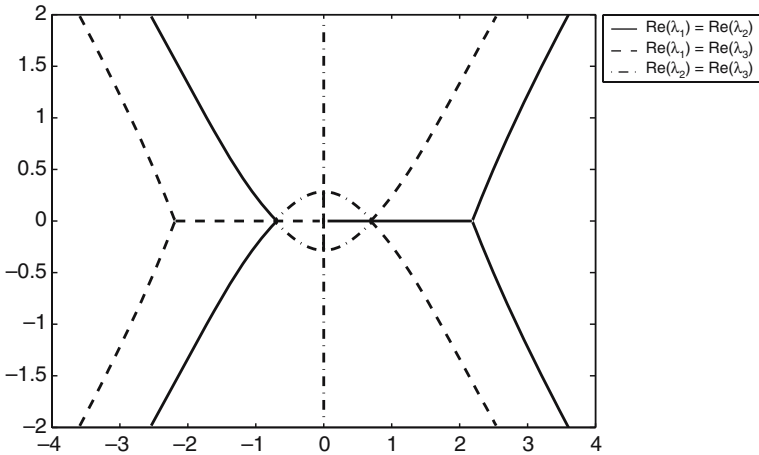


Fig. 6. Curves where $\text{Re } \lambda_j = \text{Re } \lambda_k$ for the value $a = 0.5$

in the region in the right-half plane, bounded by the solid and dashed-dotted curves, and

$$\text{Re } \lambda_3(z) > \max(\text{Re } \lambda_1(z), \text{Re } \lambda_2(z))$$

in the region in the left-half plane bounded by the dashed and dashed-dotted curves.

The exceptional neighborhood will not cause a problem to us, since it turns out to shrink fast enough if $a = 1 + (b/2)n^{-1/2}$ and $n \rightarrow \infty$. For n large enough, the exceptional neighborhood is well within the disk around the origin of radius $n^{-1/4}$ where we are going to construct a special parametrix with Pearcey integrals. Then the different ordering of the real parts of the λ_k will not play a role.

5. First Two Transformations of the RH Problem

The first and second transformation of the RH problem are the same as in our earlier paper [7], except that we use the λ -functions that were introduced in the last section via the modified ξ -functions.

5.1. First transformation $Y \mapsto T$. Using the functions λ_k and the constants $\ell_k, k = 1, 2, 3$, we define

$$T(z) = \text{diag} \left(e^{-n\ell_1}, e^{-n\ell_2}, e^{-n\ell_3} \right) Y(z) \text{diag} \left(e^{n(\lambda_1(z) - \frac{1}{2}z^2)}, e^{n(\lambda_2(z) - az)}, e^{n(\lambda_3(z) + az)} \right). \tag{5.1}$$

Then by (2.1) and (5.1) and the jump properties (4.3) we have $T_+(x) = T_-(x)j_T(x)$ for $x \in \mathbb{R}$, where

$$j_T = \begin{pmatrix} e^{n(\lambda_1-\lambda_2)_+} & 1 & e^{n(\lambda_3-\lambda_1-)} \\ 0 & e^{n(\lambda_1-\lambda_2)-} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (0, z^*), \tag{5.2}$$

$$j_T = \begin{pmatrix} e^{n(\lambda_1-\lambda_3)_+} & e^{n(\lambda_2+-\lambda_1-)} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_1-\lambda_3)-} \end{pmatrix}, \quad x \in (-z^*, 0), \tag{5.3}$$

$$j_T = \begin{pmatrix} 1 & e^{n(\lambda_2+-\lambda_1-)} & e^{n(\lambda_3+-\lambda_1-)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (-\infty, -z^*) \cup (z^*, \infty). \tag{5.4}$$

The function $T(z)$ solves the following RH problem:

- T is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- for $x \in \mathbb{R}$, we have

$$T_+(x) = T_-(x)j_T(x), \tag{5.5}$$

where j_T is given by (5.2)–(5.4),

- as $z \rightarrow \infty$,

$$T(z) = I + O\left(\frac{1}{z}\right). \tag{5.6}$$

The asymptotic property (5.6) follows from (2.2), (5.1), and the behavior (4.2) of the λ -functions at infinity.

5.2. *Second transformation $T \mapsto S$.* The second transformation of the RH problem consists of opening of lenses around the intervals $[0, z^*]$ and $[-z^*, 0]$. The lenses are as shown in Fig. 7. We define (see also Sect. 5 in [7])

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ -e^{n(\lambda_1-\lambda_2)} & 1 & -e^{n(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{in the upper right lens region,} \tag{5.7}$$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)} & 1 & -e^{n(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{in the lower right lens region,} \tag{5.8}$$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{n(\lambda_1-\lambda_3)} & -e^{n(\lambda_2-\lambda_3)} & 1 \end{pmatrix} \quad \text{in the upper left lens region,} \tag{5.9}$$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1-\lambda_3)} & -e^{n(\lambda_2-\lambda_3)} & 1 \end{pmatrix} \quad \text{in the lower left lens region,} \tag{5.10}$$

and $S = T$ outside the lenses.

It leads to a matrix valued function S which is defined and analytic in $\mathbb{C} \setminus \Sigma_S$, where Σ_S consists of the real line and the upper and lower lips of the lenses. On Σ_S we have

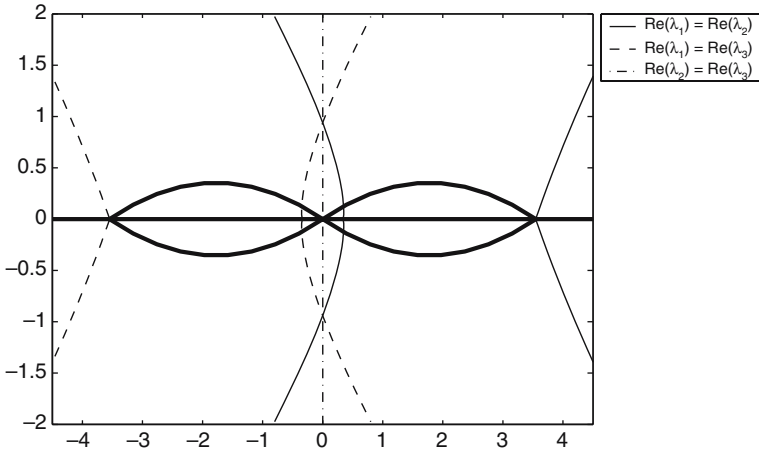


Fig. 7. Opening of lenses around the intervals $[0, z^*]$ and $[-z^*, 0]$ for the value $a = 2.0$. The upper and lower lips of the lenses together with the real axis form the contour Σ_S which is shown in bold. Also shown are the critical trajectories as in Fig. 4

$S_+ = S_- j_S$ where j_S is defined as follows (the orientation on Σ_S is taken from left to right):

$$j_S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{on } (0, z^*), \tag{5.11}$$

$$j_S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{on } (-z^*, 0), \tag{5.12}$$

$$j_S = \begin{pmatrix} 1 & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{on } (-\infty, -z^*) \cup (z^*, \infty), \tag{5.13}$$

$$j_S = \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{on the upper lip of the right lens,} \tag{5.14}$$

$$j_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1 - \lambda_3)} & e^{n(\lambda_2 - \lambda_3)} & 1 \end{pmatrix}, \quad \text{on the upper lip of the left lens,} \tag{5.15}$$

$$j_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1 - \lambda_3)} & -e^{n(\lambda_2 - \lambda_3)} & 1 \end{pmatrix}, \quad \text{on the lower lip of the left lens,} \tag{5.16}$$

$$j_S = \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & -e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{on the lower lip of the right lens.} \tag{5.17}$$

Thus S solves the following RH problem:

- S is analytic on $\mathbb{C} \setminus \Sigma_S$,

- for $z \in \Sigma_S$, we have $S_+(z) = S_-(z)j_S(z)$, where j_S is given by (5.11)–(5.17),
- as $z \rightarrow \infty$, we have $S(z) = I + O\left(\frac{1}{z}\right)$.

Now the ordering of the real parts of the λ_k in various regions in the complex plane (see Lemma 4.2) shows that the jump matrices in (5.13)–(5.17) are all close to the identity matrix if n is large, except in a neighborhood of the origin. To be precise, if $a < 1$ then $\text{Re } \lambda_3 > \text{Re } \lambda_2$ near the origin in the right half-plane, which means that the entries $\pm e^{n(\lambda_3 - \lambda_2)}$ in the jump matrices in (5.14) and (5.17) are not small near the origin but instead grow exponentially if n gets large. Similarly the entries $\pm e^{n(\lambda_2 - \lambda_3)}$ in the jump matrices in (5.15) and (5.16) also grow exponentially near the origin. On the other hand, if $a > 1$, then $\text{Re } \lambda_1$ is bigger than the other two in the exceptional neighborhood of the origin, so that the other non-zero off-diagonal entries in the jump matrices in (5.14)–(5.17) grow exponentially in a neighborhood of the origin. For $a = 1$ there are no such exceptions and all jump matrices in (5.13)–(5.17) are close to the identity matrix if n is large.

When we wrote that certain entries grow exponentially as n gets large, it was understood that the value of $a \neq 1$ remained fixed. However, eventually we are going to take $a = 1 + O(n^{-1/2})$ as $n \rightarrow \infty$. Then it will turn out that the possible growth of certain entries in the jump matrices is confined to a small enough region near the origin, which shrinks sufficiently fast as $n \rightarrow \infty$, so that we can still ignore the jumps (5.13)–(5.17) in the next step.

6. Model RH Problem

We consider the following auxiliary model RH problem: find $M : \mathbb{C} \setminus [-z^*, z^*] \rightarrow \mathbb{C}^{3 \times 3}$ such that

- M is analytic on $\mathbb{C} \setminus [-z^*, z^*]$,
- for $x \in (-z^*, z^*)$ we have $M_+(x) = M_-(x)j_M(x)$, where

$$j_M(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } x \in (0, z^*), \tag{6.1}$$

and

$$j_M(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{for } x \in (-z^*, 0), \tag{6.2}$$

- as $z \rightarrow \infty$,

$$M(z) = I + O\left(\frac{1}{z}\right). \tag{6.3}$$

This RH problem has a solution, see [7, Sect. 6], that can be explicitly given in terms of the mapping functions $w_k, k = 1, 2, 3$, from (3.1) and (3.2). The solution takes the form

$$M(z) = \begin{pmatrix} M_1(w_1(z)) & M_1(w_2(z)) & M_1(w_3(z)) \\ M_2(w_1(z)) & M_2(w_2(z)) & M_2(w_3(z)) \\ M_3(w_1(z)) & M_3(w_2(z)) & M_3(w_3(z)) \end{pmatrix}, \tag{6.4}$$

where M_1, M_2, M_3 are the three scalar valued functions

$$\begin{aligned}
 M_1(w) &= \frac{w^2 - c^2}{w\sqrt{w^2 - 3c^2}}, & M_2(w) &= \frac{-i}{\sqrt{2}} \frac{w + c}{w\sqrt{w^2 - 3c^2}}, \\
 M_3(w) &= \frac{-i}{\sqrt{2}} \frac{w - c}{w\sqrt{w^2 - 3c^2}}.
 \end{aligned}
 \tag{6.5}$$

Note that by (3.2) we have that $w_k(z)$ is of order $z^{1/3}$ as $z \rightarrow 0$. By (6.4) and (6.5) this implies that

$$M(z) = O(z^{-1/3}) \quad \text{as } z \rightarrow 0.
 \tag{6.6}$$

The RH problem for M easily gives that $\det M(z) \equiv 1$. Thus $M^{-1}(z)$ exists for $z \in \mathbb{C} \setminus [-z^*, z^*]$ and from (6.6) it follows that $M^{-1}(z) = O(z^{-2/3})$. However, the special form of the solution (6.4)–(6.5) shows that all cofactors of M are actually $O(z^{-1/3})$ as $z \rightarrow 0$. Thus

$$M^{-1}(z) = O(z^{-1/3}) \quad \text{as } z \rightarrow 0.
 \tag{6.7}$$

This may also be understood from the fact that $M^{-1} = M^t$, since together with M it is easy to see that also M^{-t} is a solution of the RH problem (6.1)–(6.3).

The model solution M will be used to construct a *parametrix* for S outside of small neighborhoods of the edge points and the origin. Namely, we consider disks of fixed radius r around the edge points and a shrinking disk $D(0, n^{-1/4})$ of radius $n^{-1/4}$ around the origin. At the edge points and at the origin M is not analytic (it is not even bounded) and in the disks around the edge points and the origin the parametrix is constructed differently.

7. Parametrix at Edge Points

The construction of a parametrix P at the edge points $\pm z^*$ can be done with Airy functions in a by now standard way, see [6, 15–17]. We omit details. We only note that $\pm z^* = \pm \frac{3\sqrt{3}}{2}c$ depends on c and therefore on a . As $a \rightarrow 1$ we have $c \rightarrow 1$ and so $\pm z^* \rightarrow \pm \frac{3\sqrt{3}}{2}$. We construct the Airy parametrices in fixed neighborhoods $D(\pm \frac{3\sqrt{3}}{2}, r)$ of $\pm \frac{3\sqrt{3}}{2}$ so that

- P is analytic on $D(\pm \frac{3\sqrt{3}}{2}, r) \setminus \Sigma_S$,
- for $z \in D(\pm \frac{3\sqrt{3}}{2}, r) \cap \Sigma_S$, we have

$$P_+(z) = P_-(z)j_S(z),
 \tag{7.1}$$

where j_S is given by (5.11)–(5.17),

- as $n \rightarrow \infty$,

$$P(z) = M(z) \left(I + O\left(n^{-1}\right) \right) \quad \text{uniformly for } \left| z \pm \frac{3\sqrt{3}}{2} \right| = r.
 \tag{7.2}$$

8. Parametrix at the Origin

The main issue is the construction of a parametrix at the origin and this is where the Pearcey integrals come in. For a sufficiently close to 1, we want to define Q in a neighborhood $D(0, r)$ of the origin such that

- Q is analytic on $D(0, r) \setminus \Sigma_S$,
- for $z \in D(0, r) \cap \Sigma_S$, we have

$$Q_+(z) = Q_-(z)j_S(z), \tag{8.1}$$

where j_S is given by (5.11)–(5.17),

- as $n \rightarrow \infty$, and with $a = 1 + O(n^{-1/2})$, we have

$$Q(z) = M(z) \left(I + O \left(n^{-1/2} \right) \right) \quad \text{uniformly for } |z| = n^{-1/4}. \tag{8.2}$$

The parametrix Q will be constructed with the aid of Pearcey integrals.

To motivate the construction, we note that the jump matrices for S can be factored as

$$j_S = e^{-n\Lambda_-} j_S^o e^{n\Lambda_+}, \tag{8.3}$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and

$$j_S^o = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (0, z^*), \tag{8.4}$$

$$j_S^o = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{on } (-z^*, 0), \tag{8.5}$$

$$j_S^o = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the upper lip of the right lens,} \tag{8.6}$$

$$j_S^o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{on the upper lip of the left lens,} \tag{8.7}$$

$$j_S^o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{on the lower lip of the left lens,} \tag{8.8}$$

$$j_S^o = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the lower lip of the right lens.} \tag{8.9}$$

We show in the next subsection that the Pearcey integrals satisfy a RH problem with exactly the above jump matrices except that these jumps are situated on six rays emanating from the origin.

8.1. The Pearcey parametrix. Let $b \in \mathbb{R}$ be fixed. The Pearcey differential equation $p'''(\zeta) = \zeta p(\zeta) + bp'(\zeta)$ admits solutions of the form

$$p_j(\zeta) = \int_{\Gamma_j} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2 + is\zeta} ds \tag{8.10}$$

for $j = 0, 1, 2, 3, 4, 5$, where

$$\begin{aligned} \Gamma_0 &= (-\infty, \infty), & \Gamma_1 &= (i\infty, 0] \cup [0, \infty), \\ \Gamma_2 &= (i\infty, 0] \cup [0, -\infty), & \Gamma_3 &= (-i\infty, 0] \cup [0, -\infty), \\ \Gamma_4 &= (-i\infty, 0] \cup [0, \infty), & \Gamma_5 &= (-i\infty, i\infty), \end{aligned} \tag{8.11}$$

or any other contours that are homotopic to them as for example given in Fig. 8. The formulas (8.11) also determine the orientation of the contours Γ_j .

Define $\Phi = \Phi(\zeta; b)$ in six sectors by

$$\Phi = \begin{pmatrix} -p_2 & p_1 & p_5 \\ -p_2' & p_1' & p_5' \\ -p_2'' & p_1'' & p_5'' \end{pmatrix} \quad \text{for } 0 < \arg \zeta < \pi/4, \tag{8.12}$$

$$\Phi = \begin{pmatrix} p_0 & p_1 & p_4 \\ p_0' & p_1' & p_4' \\ p_0'' & p_1'' & p_4'' \end{pmatrix} \quad \text{for } \pi/4 < \arg \zeta < 3\pi/4, \tag{8.13}$$

$$\Phi = \begin{pmatrix} -p_3 & -p_5 & p_4 \\ -p_3' & -p_5' & p_4' \\ -p_3'' & -p_5'' & p_4'' \end{pmatrix} \quad \text{for } 3\pi/4 < \arg \zeta < \pi, \tag{8.14}$$

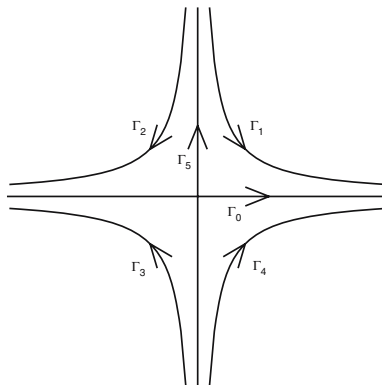


Fig. 8. The contours $\Gamma_j, j = 0, 1, \dots, 5$, equivalent to those in (8.11), that are used in the definition of the Pearcey integrals p_j

$$\Phi = \begin{pmatrix} p_4 & -p_5 & p_3 \\ p'_4 & -p'_5 & p'_3 \\ p''_4 & -p''_5 & p''_3 \end{pmatrix} \quad \text{for } -\pi < \arg \zeta < -3\pi/4, \quad (8.15)$$

$$\Phi = \begin{pmatrix} p_0 & p_2 & p_3 \\ p'_0 & p'_2 & p'_3 \\ p''_0 & p''_2 & p''_3 \end{pmatrix} \quad \text{for } -3\pi/4 < \arg \zeta < -\pi/4, \quad (8.16)$$

$$\Phi = \begin{pmatrix} p_1 & p_2 & p_5 \\ p'_1 & p'_2 & p'_5 \\ p''_1 & p''_2 & p''_5 \end{pmatrix} \quad \text{for } -\pi/4 < \arg \zeta < 0. \quad (8.17)$$

Then Φ has jumps on the six rays. We choose an orientation on these rays so that the rays in the right half-plane are oriented from 0 to ∞ , and the rays in the left half-plane are oriented from ∞ to 0.

Then the integral representations (8.10)–(8.11) easily imply that $\Phi_+ = \Phi_- j_\Phi$, where

$$j_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \arg \zeta = \pi/4, \quad j_\Phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \arg \zeta = 0, \quad (8.18)$$

$$j_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \arg \zeta = -\pi/4, \quad j_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{on } \arg \zeta = 3\pi/4, \quad (8.19)$$

$$j_\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{on } \arg \zeta = -\pi, \quad j_\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{on } \arg \zeta = -3\pi/4. \quad (8.20)$$

So these are indeed the jump matrices of (8.4)–(8.9).

8.2. Asymptotics of Pearcey integrals. A classical steepest descent analysis of the integral representations gives the following result for the asymptotic behavior of $\Phi(\zeta; b)$ as $\zeta \rightarrow \infty$. As always we use the principal branches of the fractional powers, that is, with a branch cut along the negative axis.

Lemma 8.1. *For every fixed $b \in \mathbb{C}$, we have as $\zeta \rightarrow \infty$,*

$$\begin{aligned} \Phi(\zeta; b) &= \sqrt{\frac{2\pi}{3}} i e^{\frac{b^2}{8}} \begin{pmatrix} \zeta^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{1/3} \end{pmatrix} \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix} \\ &\times \left(I + O\left(\zeta^{-2/3}\right) \right) \begin{pmatrix} e^{\theta_1(\zeta; b)} & 0 & 0 \\ 0 & e^{\theta_2(\zeta; b)} & 0 \\ 0 & 0 & e^{\theta_3(\zeta; b)} \end{pmatrix} \end{aligned} \quad (8.21)$$

for $\text{Im } \zeta > 0$, and

$$\begin{aligned} \Phi(\zeta; b) &= \sqrt{\frac{2\pi}{3}} i e^{\frac{b^2}{8}} \begin{pmatrix} \zeta^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{1/3} \end{pmatrix} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix} \\ &\times \left(I + O\left(\zeta^{-2/3}\right) \right) \begin{pmatrix} e^{\theta_2(\zeta; b)} & 0 & 0 \\ 0 & e^{\theta_1(\zeta; b)} & 0 \\ 0 & 0 & e^{\theta_3(\zeta; b)} \end{pmatrix} \end{aligned} \quad (8.22)$$

for $\text{Im } \zeta < 0$, where $\omega = e^{2\pi i/3}$ and

$$\theta_k(\zeta; b) = \frac{3}{4}\omega^{2k}\zeta^{4/3} + \frac{b}{2}\omega^k\zeta^{2/3}, \quad k = 1, 2, 3. \tag{8.23}$$

The O -terms in (8.21) and (8.22) are uniform for b in a bounded subset of the complex plane.

Proof. We give an outline of the proof, cf. also the calculations in [19]. Let $\theta(s; \zeta, b) = -\frac{1}{4}s^4 - \frac{b}{2}s^2 + i\zeta s$. The saddle point equation for (8.10) is

$$\frac{\partial \theta}{\partial s} = -s^3 - bs + i\zeta = 0.$$

For $b = 0$ there are three solutions $s_k^o = -i\omega^k\zeta^{1/3}$, $k = 1, 2, 3$, and as $\zeta \rightarrow \infty$, while b remains bounded, the three saddles $s_k = s_k(\zeta; b)$ are close to s_k^o , and in fact

$$s_k(\zeta; b) = -i\omega^k\zeta^{1/3} - i\omega^{2k}\frac{b}{3}\zeta^{-1/3} + O(\zeta^{-5/3}) \quad \text{as } \zeta \rightarrow \infty.$$

The value at the saddles is

$$\theta(s_k(\zeta; b); \zeta, b) = \frac{3}{4}\omega^k\zeta^{4/3} + \frac{b}{2}\omega^{2k}\zeta^{2/3} + \frac{b^2}{6} + O(\zeta^{-2/3}) \quad \text{as } \zeta \rightarrow \infty.$$

Then, if C_k is the steepest descent path through s_k , we obtain from classical steepest descent arguments

$$\begin{aligned} \int_{C_k} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2 + i\zeta s} ds &= \pm \sqrt{\frac{2\pi}{-\frac{\partial^2 \theta}{\partial s^2}(s_k(\zeta, b); \zeta, b)}} e^{\theta(s_k(\zeta; b); \zeta, b)} (1 + O(\zeta^{-2/3})) \\ &= \pm \sqrt{\frac{2\pi}{3}} i\omega^{2k}\zeta^{-1/3} e^{\frac{3}{4}\omega^k\zeta^{4/3} + \frac{b}{2}\omega^{2k}\zeta^{2/3} + \frac{b^2}{6}} (1 + O(\zeta^{-2/3})). \end{aligned}$$

The choice of \pm sign depends on the orientation of the steepest descent path.

Now take any of the six sectors that appear in the definition (8.12)–(8.17) of Φ and take some p_j that appears in the definition of Φ in that sector. The contour Γ_j in the definition (8.10) of p_j can be deformed to the steepest descent contour through one of the saddles, or to the union of two or three such steepest descent contours. However, in the latter case, it turns out that there is always a unique dominant saddle for p_j in that particular sector. Thus for some k and some choice of \pm sign, we have

$$p_j(\zeta) = \pm \sqrt{\frac{2\pi}{3}} i\omega^{2k}\zeta^{-1/3} e^{\frac{3}{4}\omega^k\zeta^{4/3} + \frac{b}{2}\omega^{2k}\zeta^{2/3} + \frac{b^2}{6}} (1 + O(\zeta^{-2/3})) \tag{8.24}$$

as $\zeta \rightarrow \infty$ in the chosen sector. Similarly,

$$p'_j(\zeta) = \pm \sqrt{\frac{2\pi}{3}} i e^{\frac{3}{4}\omega^k\zeta^{4/3} + \frac{b}{2}\omega^{2k}\zeta^{2/3} + \frac{b^2}{6}} (1 + O(\zeta^{-2/3})), \tag{8.25}$$

$$p''_j(\zeta) = \pm \sqrt{\frac{2\pi}{3}} i\omega^k\zeta^{1/3} e^{\frac{3}{4}\omega^k\zeta^{4/3} + \frac{b}{2}\omega^{2k}\zeta^{2/3} + \frac{b^2}{6}} (1 + O(\zeta^{-2/3})). \tag{8.26}$$

A further analysis reveals which value of k and what sign is associated with p_j in the particular sector. We will not go through this analysis here, but the result is given by (8.21) and (8.22). This completes the proof of the lemma. \square

Note that in the above lemma we only state the leading term in a full asymptotic expansion, which is enough for the purposes of this paper. We also stay away from situations where saddles coalesce. For more asymptotic results on Pearcey integrals in various regimes, see [4, 19, 21] and the references cited therein.

8.3. *Definition of Q.* We are going to define the local parametrix Q in the form

$$Q(z) = E(z)\Phi(n^{3/4}\zeta(z); n^{1/2}b(z))e^{n\Lambda(z)}e^{-nz^2/6}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3), \quad (8.27)$$

where E is an analytic prefactor, $z \mapsto \zeta(z)$ is a conformal map from a neighborhood of 0 in the z -plane to a neighborhood of 0 in the ζ -plane, and $z \mapsto b(z)$ is analytic.

We choose $\zeta(z)$ and $b(z)$ so that the exponential factors in the asymptotic behavior of $\Phi(n^{3/4}\zeta(z); n^{1/2}b(z))$ are cancelled when we multiply them by $e^{n\Lambda(z)}e^{-nz^2/6}$. We use the functions f_3 and g_3 from Lemma 4.1 in the following definition. These functions depend on a , and to emphasize the a -dependence we write $f_3(z; a)$ and $g_3(z; a)$. The functions $\zeta(z)$ and $b(z)$ also depend on a .

Definition. For z in a sufficiently small neighborhood of 0, we define

$$\zeta(z) = \zeta(z; a) = z [f_3(z; a)]^{3/4}, \quad (8.28)$$

and

$$b(z) = b(z; a) = \frac{g_3(z; a)}{f_3(z; a)^{1/2}}. \quad (8.29)$$

In (8.28) and (8.29) the branch of the fractional powers is chosen which is real and positive for real values of z near 0.

Lemma 8.2. (a) *There is an $r > 0$ and a $\delta > 0$ so that for each $a \in (1 - \delta, 1 + \delta)$ we have that $z \mapsto \zeta(z; a)$ is a conformal map on the disk $D(0, r)$ and $z \mapsto b(z; a)$ is analytic on $D(0, r)$.*

(b) *In addition we have*

$$b(z; a) = O(a - 1) + O(z^2) \quad \text{as } a \rightarrow 1 \text{ and } z \rightarrow 0. \quad (8.30)$$

Proof. Following the constructions of f_j and g_j for $j = 1, 2, 3$ in Lemmas 3.2, 3.3, and 4.1 and their proofs, we easily see that

$$f_3(z; a) = f_3(z; 1) + O(a - 1), \quad g_3(z; a) = g_3(z; 1) + O(a - 1), \quad \text{as } a \rightarrow 1, \quad (8.31)$$

uniformly for z in a neighborhood of 0, and

$$f_3(z; 1) = 1 + O(z^2), \quad g_3(z; 1) = O(z^2) \quad \text{as } z \rightarrow 0. \quad (8.32)$$

Both parts of the lemma follow from (8.31) and (8.32), and the definitions (8.28) and (8.29). \square

From now on we assume that $|a - 1| < \delta$, where $\delta > 0$ is as in part (a) of Lemma 8.2, so that $z \mapsto \zeta(z; a)$ is a conformal map. Near 0 we choose the precise form of the lenses so that the lips of the lenses are mapped by $z \mapsto \zeta(z; a)$ to the rays $\arg \zeta = \pm\pi/4$ and $\arg \zeta = \pm 3\pi/4$. Then from the fact that the jump matrices (8.18)–(8.20) of Φ agree with those in (8.4)–(8.9), it follows that the jump condition (8.1) for Q is satisfied. This holds for any choice of analytic prefactor E that is used in (8.27) to define Q . We are going to define E so that the matching condition (8.2) is satisfied as well.

8.4. *Matching condition.* To obtain the matching condition (8.2) we first note that the definitions (8.28) and (8.29) give us (we drop the a -dependence in the notation)

$$\zeta(z)^{4/3} = z^{4/3} f_3(z), \quad b(z)\zeta(z)^{2/3} = g_3(z).$$

Hence by (4.4) and (8.23) we have for $\text{Im } z > 0$ with $|z| < r$,

$$\theta_k(\zeta(z); b(z)) + \lambda_k(z) - z^2/6 = 0, \quad k = 1, 2, 3, \tag{8.33}$$

while for $\text{Im } z < 0$ with $|z| < r$,

$$\begin{aligned} \theta_2(\zeta(z); b(z)) + \lambda_1(z) - z^2/6 &= \lambda_{1-}(0) = \pi i, \\ \theta_1(\zeta(z); b(z)) + \lambda_2(z) - z^2/6 &= \lambda_{2-}(0) = -\pi i, \\ \theta_3(\zeta(z); b(z)) + \lambda_3(z) - z^2/6 &= \lambda_{3-}(0) = 0. \end{aligned} \tag{8.34}$$

Assume $a = 1 + O(n^{-1/2})$. Then it follows from (8.30) that

$$n^{1/2}|b(z; a)| \leq C \quad \text{for } |z| \leq 2n^{-1/4} \tag{8.35}$$

for every n large enough, with a value C that is independent of n . As a consequence we can use the expansions (8.21), (8.22) as $n \rightarrow \infty$, because of Lemma 8.1. We find from (8.21) and (8.22) and the relations (8.33) and (8.34) between θ_k and λ_k that the exponential factors in the asymptotic behavior (as $n \rightarrow \infty$) of

$$\Phi(n^{3/4}\zeta(z); n^{1/2}b(z))e^{n\Lambda}e^{-nz^2/6}$$

cancel if we take z so that $0.9n^{-1/4} \leq |z| \leq 1.1n^{-1/4}$. So we have proved the following.

Lemma 8.3. *Let $a = 1 + O(n^{-1/2})$. Then we have as $n \rightarrow \infty$, uniformly for z so that $0.9n^{-1/4} \leq |z| \leq 1.1n^{-1/4}$, that*

$$\begin{aligned} Q(z) &= E(z)\Phi(n^{3/4}\zeta(z); n^{1/2}b(z))e^{n\Lambda(z)}e^{-nz^2/6} \\ &= \sqrt{\frac{2\pi}{3}}ie^{nb(z)^2/8}E(z) \begin{pmatrix} n^{-1/4} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & n^{1/4} \end{pmatrix} K(\zeta(z))(I + O(n^{-1/3})), \end{aligned} \tag{8.36}$$

where

$$K(\zeta) = \begin{cases} \begin{pmatrix} \zeta^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{1/3} \end{pmatrix} \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix} & \text{for } \text{Im } \zeta > 0, \\ \begin{pmatrix} \zeta^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{1/3} \end{pmatrix} \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix} & \text{for } \text{Im } \zeta < 0. \end{cases} \tag{8.37}$$

Proof. This follows from the asymptotic behavior (8.21) and (8.22), since we have shown in the above that the exponential factors in (8.21) and (8.22) are cancelled when we multiply them by $e^{n\Lambda(z)}e^{-nz^2/6}$.

As for the O -term, we note that $n^{3/4}\zeta(z) = O(n^{1/2})$ if $|z| = cn^{-1/4}$ with $0.9 \leq c \leq 1.1$, so that the $O(\zeta^{-2/3})$ term in (8.21)–(8.22) leads to the $O(n^{-1/3})$ term in (8.36). \square

In order to achieve the matching (8.2) of $Q(z)$ with $M(z)$ we now *define* the prefactor E by

$$E(z) = -\sqrt{\frac{3}{2\pi}} i e^{-nb(z)^2/8} M(z)K(\zeta(z))^{-1} \begin{pmatrix} n^{1/4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-1/4} \end{pmatrix}. \tag{8.38}$$

Then the matching condition (8.2) follows from (8.36) and (8.38).

It only remains to check that E is analytic in a full neighborhood of the origin. This follows since M and K satisfy the same jump relations on the real line. Indeed we have from the expressions (8.37) for K , for real ζ with $\zeta > 0$,

$$K_-(\zeta)^{-1}K_+(\zeta) = \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

while for real $\zeta < 0$ we have to take into account that $\zeta^{1/3}$ and $\zeta^{-1/3}$ have different \pm -boundary values, so that for $\zeta < 0$,

$$\begin{aligned} K_-(\zeta)^{-1}K_+(\zeta) &= \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \zeta_-^{1/3} \zeta_+^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_-^{-1/3} \zeta_+^{1/3} \end{pmatrix} \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix} \\ &= \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega^2 & \omega & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

These are indeed equal to the jumps satisfied by M ; see (6.1) and (6.2). Since $\zeta(z)$ is a conformal map on $D(0, r)$ that is real and positive for $z \in (0, r)$, and real and negative for $z \in (-r, 0)$, we find that $M(z)K(\zeta(z))^{-1}$ is analytic across both $(0, r)$ and $(-r, 0)$. Thus $E(z)$ is analytic in $D(0, r) \setminus \{0\}$. The isolated singularity at 0 is removable, since the entries in $M(z)$ and $K(\zeta(z))^{-1}$ have at most $z^{-1/3}$ -type singularity at the origin, and they cannot combine to form a pole. The conclusion is that E is analytic.

This completes the construction of the local parametrix Q at the origin.

9. Final Transformation

We now fix $b \in \mathbb{R}$ and let $a = 1 + \frac{b}{2\sqrt{n}}$. Now we define

$$R(z) = \begin{cases} S(z)M(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \Sigma_S \text{ outside the disks } D(0, n^{-1/4}) \text{ and } D(\pm \frac{3\sqrt{3}}{2}, r), \\ S(z)P(z)^{-1}, & \text{for } z \in D(\pm \frac{3\sqrt{3}}{2}, r) \setminus \Sigma_S, \\ S(z)Q(z)^{-1}, & \text{for } z \in D(0, n^{-1/4}) \setminus \Sigma_S. \end{cases} \tag{9.1}$$

Then R is analytic inside the disks and also across the real interval between the disks. Thus R is analytic outside the contour Σ_R shown in Fig. 9.

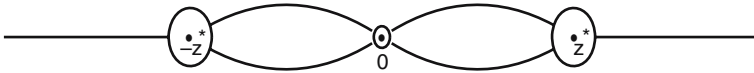


Fig. 9. The contour Σ_R . The matrix-valued function R is analytic on $\mathbb{C} \setminus \Sigma_R$. The disk around 0 has radius $n^{-1/4}$ and is shrinking as $n \rightarrow \infty$. The disks are oriented counterclockwise and the remaining parts of Σ_R are oriented from left to right

Lemma 9.1. *We have $R_+ = R_- j_R$ where*

$$j_R(z) = I + O(n^{-1}) \quad \text{uniformly for } \left| z \mp \frac{3\sqrt{3}}{2} \right| = r, \tag{9.2}$$

$$j_R(z) = I + O(n^{-1/6}) \quad \text{uniformly for } |z| = n^{-1/4}, \tag{9.3}$$

and there exists $c > 0$ so that

$$j_R(z) = I + O\left(\frac{e^{-cn^{2/3}}}{1+|z|^2}\right) \quad \text{uniformly for } z \text{ on the remaining parts of } \Sigma_R. \tag{9.4}$$

Proof. The behavior (9.2) of the jump matrix on the circles around the endpoints $\pm \frac{3\sqrt{3}}{2}$ is a result of the construction of the Airy parametrix. It follows as in [16, 17].

The jump matrix for $|z| = n^{-1/4}$ is by (9.1) and (8.2) (we use positive orientation)

$$j_R = M Q - 1 = M(I + O(n^{-1/3}))M^{-1} = M + M O(n^{-1/3})M^{-1}(z).$$

Since $M(z) = O(z^{-1/3})$ and $M^{-1}(z) = O(z^{-1/3})$ as $z \rightarrow 0$ by (6.6) and (6.7), we obtain (9.3).

The jump matrix $j_R(z)$ on the remaining part of Σ_R is $I + O(e^{-cn})$ if $z \in \Sigma_R$ stays at a fixed distance of 0 and $\pm \frac{3\sqrt{3}}{2}$. But now the disk around 0 is shrinking as n increases, and so we have to be more careful here. We note that the jump matrix is

$$j_R(z) = M(z)j_S(z)M^{-1}(z)$$

and we want to know its behavior as $n \rightarrow \infty$ for z on the lips of the lenses near 0 and $|z| \geq n^{-1/4}$.

The jump matrices j_S in (5.14)–(5.17) contain off-diagonal entries $\pm e^{n(\lambda_k - \lambda_j)}$. For $a = 1$ these entries are decaying on the contours and so we have for some positive constant $c_1 > 0$,

$$\text{Re}((\lambda_j - \lambda_k)(z; 1)) \geq c_1 |z|^{4/3}$$

for z on the lips of the lenses near 0. Since $\lambda_j(z; a) = \lambda_j(z; 1) + z^{2/3} O(a - 1)$ as $a \rightarrow 1$, we then get that

$$\text{Re}((\lambda_j - \lambda_k)(z; a)) \geq c_1 z^{-4/3} - c_2 |z|^{2/3} |a - 1|.$$

Then if $a - 1 = (b/2)n^{-1/2}$ and $|z| \geq n^{-1/4}$ we easily get that

$$\text{Re}(\lambda_j - \lambda_k)(z; a) \geq c_3 n^{-1/3}$$

for some positive constant $c_3 > 0$. Then it follows from (5.14)–(5.17) that

$$j_S(z) = I + O(e^{-c_3 n^{2/3}}).$$

This leads to (9.4) since $M(z) = O(z^{-1/3})$ and $M^{-1}(z) = O(z^{-1/3})$ as $z \rightarrow 0$, see (6.6) and (6.7). \square

To summarize, we find that R solves the following RH problem:

- R is analytic on $\mathbb{C} \setminus \Sigma_R$,
- for $z \in \Sigma_R$, we have $R_+ = R_- j_R$, where j_R satisfies (9.2)–(9.4),
- as $z \rightarrow \infty$, we have $R(z) = I + O(1/z)$.

The RH problem for R is posed on a contour that is varying with n . This is a slight complication. However we still can guarantee the following behavior of R as $n \rightarrow \infty$.

Proposition 9.2. *As $n \rightarrow \infty$ we have that*

$$R(z) = I + O\left(\frac{n^{-1/6}}{1 + |z|}\right) \tag{9.5}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

Since the proof of Proposition 9.2 is somewhat technical due to the fact that the contours are varying with n , we give it in Appendix A.

10. Proof of Theorem 1.1

Now we are ready for the proof of Theorem 1.1. We fix $b \in \mathbb{R}$ and take

$$a = 1 + \frac{b}{2\sqrt{n}}.$$

10.1. The effect of the transformations $Y \mapsto T \mapsto S \mapsto R$. We are going to follow the effect of the transformations on the correlation kernel $K_n(x, y; a)$ for real values of x and y close to 0. We start from (2.3) which gives $K_n(x, y; a)$ in terms of the solution of the RH problem for Y . The transformation (5.1) then implies that

$$K_n(x, y; a) = \frac{e^{\frac{1}{4}n(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} 0 & e^{n\lambda_{2+}(y)} & e^{n\lambda_{3+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix}. \tag{10.1}$$

According to the transformation $T \mapsto S$ given in (5.7)–(5.10) we now have to distinguish between x and y being positive or negative. We will do the calculations explicitly for $x > 0$ and $y < 0$. The other cases are treated in the same way.

So we assume that $x > 0$ and $y < 0$, and both of them are close to 0. The formulas (5.7) and (5.9) applied to (10.1) then give

$$K_n(x, y; a) = \frac{e^{\frac{1}{4}n(x^2 - y^2)}}{2\pi i(x - y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & 0 & e^{n\lambda_{3+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ e^{-n\lambda_{2+}(x)} \\ 0 \end{pmatrix}. \tag{10.2}$$

Next we note that for z close to 0, inside the disk of radius $n^{-1/4}$, we have by (9.1),

$$S(z) = R(z)Q(z) = R(z)\tilde{Q}(z)e^{n\Lambda(z)}e^{-nz^2/6},$$

where

$$\tilde{Q}(z) = Q(z)e^{-n\Lambda(z)}e^{nz^2/6} = E(z)\Phi(n^{3/4}\zeta(z; a); n^{1/2}b(z; a)), \tag{10.3}$$

see (8.36). Thus if $0 < x < n^{-1/4}$ and $-n^{-1/4} < y < 0$, we have

$$S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ e^{-n\lambda_{2+}(x)} \\ 0 \end{pmatrix} = R(x)\tilde{Q}(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-nx^2/6}$$

and

$$\begin{pmatrix} -e^{n\lambda_{1+}(y)} & 0 \\ 0 & e^{n\lambda_{3+}(y)} \end{pmatrix} S_+^{-1}(y) = e^{ny^2/6} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \tilde{Q}_+^{-1}(y)R^{-1}(y).$$

Inserting these two relations into (10.2) we find that

$$K_n(x, y; a) = \frac{e^{\frac{1}{12}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \tilde{Q}_+^{-1}(y)R^{-1}(y)R(x)\tilde{Q}_+(x) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad (10.4)$$

To obtain the scaling limit (1.6) of K_n we need the following lemma.

Lemma 10.1. *Let $a_n = 1 + (b/2)n^{-1/2}$.*

(a) *Let $x_n = xn^{-3/4}$, where $x \in \mathbb{R}$ is fixed. Then*

$$\lim_{n \rightarrow \infty} n^{3/4} \zeta(x_n; a_n) = x \quad (10.5)$$

and

$$\lim_{n \rightarrow \infty} n^{1/2} b(x_n; a_n) = b. \quad (10.6)$$

(b) *Let also $y_n = yn^{-3/4}$ where $y \in \mathbb{R}$ is fixed. Then*

$$\lim_{n \rightarrow \infty} E^{-1}(y_n)R^{-1}(y_n)R(x_n)E(x_n) = I \quad (10.7)$$

Proof. (a) Since $\zeta(z; a) = z [f_3(z; a)]^{3/4}$ by (8.28) and $f_3(z; a) \rightarrow 1$ if $z \rightarrow 0$ and $a \rightarrow 1$ by (8.31) and (8.32), we get that the limit (10.5) immediately follows.

For (10.6) we need to go back to the definitions in Lemmas 3.2, 3.3, and 4.1 of f_j and g_j , for $j = 1, 2, 3$. From 3.2 and its proof it follows that $f_1(z; a) = f_1(0; a) + O(z^2)$ and $g_1(z; a) = g_1(0) + O(z^2)$ as $z \rightarrow 0$, and the O -terms are uniform with respect to a in a neighborhood of 1. Then by (3.13) we have

$$g_2(z; a) = g_2(0; a) + O(z^2) \quad \text{as } z \rightarrow 0 \quad (10.8)$$

uniformly for a in a neighborhood of 1. Since we have, cf. Lemmas 3.3 and 4.1,

$$\frac{1}{2}z^{2/3}g_3(z; a) = \int_0^z s^{-1/3}g_2(s; a)ds,$$

we get from (10.8) that

$$g_3(z; a) = g_3(0; a) + O(z^2) \quad \text{as } z \rightarrow 0 \quad (10.9)$$

again uniformly for a in a neighborhood of 1. By (4.5) we have $g_3(0; a) = 3c^{-2/3}(c^2 - 1)$, where $c = (a + \sqrt{a^2 + 8})/4 = (a - 1)/3 + O((a - 1)^2)$ as $a \rightarrow 1$. Thus $g_3(0; a) = 2(a - 1) + O((a - 1)^2)$ as $a \rightarrow 1$, and it follows from (10.9) and the definitions of a_n and x_n that

$$\begin{aligned} n^{1/2}g_3(x_n; a_n) &= n^{1/2}g_3(0; a_n) + O(n^{-1}) = 2(a_n - 1) + O(n^{-1/2}) \\ &= b + O(n^{-1/2}) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Then (10.6) follows because of the definition (8.29) and the fact that $f_3(x_n; a_n) \rightarrow 1$ as $n \rightarrow \infty$.

(b) Since $M(z)K(\zeta(z))^{-1}$ is analytic in a neighborhood of the origin and $x_n - y_n = O(n^{-3/4})$, we have

$$K(\zeta(y_n))M(y_n)^{-1}M(x_n)K(\zeta(x_n))^{-1} = I + O(n^{-3/4})$$

as $n \rightarrow \infty$. Hence by (8.38),

$$\begin{aligned} E^{-1}(y_n)E(x_n) &= e^{n(b(y_n;a_n)^2 - b(x_n;a_n)^2)/8} \begin{pmatrix} n^{-1/4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{1/4} \end{pmatrix} \\ &\quad \times (I + O(n^{-3/4})) \begin{pmatrix} n^{1/4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n^{-1/4} \end{pmatrix} \\ &= e^{n(b(y_n;a_n)^2 - b(x_n;a_n)^2)/8} (I + O(n^{-1/4})). \end{aligned} \tag{10.10}$$

Note that both $nb(y_n; a_n)^2$ and $nb(x_n; a_n)^2$ tend to b^2 as $n \rightarrow \infty$ because of (10.6). Thus we also get from (8.38) that

$$E(x_n) = O(n^{1/4}), \quad \text{and} \quad E^{-1}(y_n) = O(n^{1/4}). \tag{10.11}$$

Next, we get from (9.5) and Cauchy’s theorem that for $z = O(n^{-3/4})$ we have

$$\frac{d}{dz}R(z) = \frac{1}{2\pi i} \int_{|s|=n^{-1/4}} \frac{R(s)}{s - z} ds = O(n^{-1/6}) \quad \text{as } n \rightarrow \infty.$$

Then by the mean-value theorem,

$$R(x_n) - R(y_n) = O((x_n - y_n)n^{-1/6}) = O(n^{-11/12})$$

so that

$$R^{-1}(y_n)R(x_n) = I + R^{-1}(y_n)(R(x_n) - R(y_n)) = I + O(n^{-11/12}). \tag{10.12}$$

Combining (10.6), (10.10), (10.11) and (10.12) we obtain (10.7). \square

Now we can compute the double scaling limit of K_n . Indeed, it follows from (10.3), (10.4), and Lemma 10.1 that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{3/4}} K_n \left(\frac{x}{n^{3/4}}, \frac{y}{n^{3/4}}; 1 + \frac{b}{2\sqrt{n}} \right) = K^{cusp}(x, y; b), \tag{10.13}$$

where

$$K^{cusp}(x, y; b) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \times \Phi_+^{-1}(y; b)\Phi_+(x; b) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{if } x > 0 \text{ and } y < 0. \quad (10.14)$$

Similar calculations show that the limit (10.13) exists for all x and y , and

$$K^{cusp}(x, y; b) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \times \Phi_+^{-1}(y; b)\Phi_+(x; b) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{if } x > 0 \text{ and } y > 0, \quad (10.15)$$

$$K^{cusp}(x, y; b) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \times \Phi_+^{-1}(y; b)\Phi_+(x; b) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } x < 0 \text{ and } y > 0, \quad (10.16)$$

$$K^{cusp}(x, y; b) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \times \Phi_+^{-1}(y; b)\Phi_+(x; b) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{if } x < 0 \text{ and } y < 0. \quad (10.17)$$

10.2. *Different formula for K^{cusp} .* To complete the proof of Theorem 1.1 we show that the formulas (10.14)–(10.17) for K^{cusp} can be rewritten in the form (1.7) given in the theorem. This involves the Pearcey integrals $p(x)$ and $q(y)$ of (1.8).

Define

$$\tilde{\Phi} = \begin{pmatrix} p_0 & p_1 & p_4 \\ p'_0 & p'_1 & p'_4 \\ p''_0 & p''_1 & p''_4 \end{pmatrix}. \quad (10.18)$$

Then by (8.13) we have that $\tilde{\Phi}(\zeta)$ agrees with $\Phi(\zeta)$ in the sector $\pi/4 < \arg \zeta < 3\pi/4$, but (10.18) defines $\tilde{\Phi}$ in the full complex ζ -plane, and in particular on the real axis.

Using the jump relation $\Phi_+ = \Phi_- j_\Phi$ for $\arg \zeta = \pi/4$ and $\arg \zeta = 3\pi/4$, see (8.18) and (8.19), we find that

$$\Phi_+(x; b) = \tilde{\Phi}(x; b) \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } x > 0$$

and

$$\Phi_+(x; b) = \tilde{\Phi}(x; b) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{if } x < 0.$$

Inserting this into (10.14)–(10.17) we find that all four cases lead to

$$K^{cusp}(x, y; b) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \tilde{\Phi}^{-1}(y; b) \tilde{\Phi}(x; b) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{10.19}$$

which is the same expression for all $x, y \in \mathbb{R}$.

Our next task is to compute $\tilde{\Phi}^{-1}$. The inverse of $\tilde{\Phi}$ is built out of solutions of

$$q'''(z) = -zq(z) + bq'(z). \tag{10.20}$$

It is easy to see that for any solution q of (10.20) and any solution p of the Pearcey equation

$$p'''(z) = zp(z) + bp'(z) \tag{10.21}$$

have $(pq'' - p'q' + p''q - bpq)' = 0$ so that

$$[p, q] := pq'' - p'q' + p''q - bpq = \text{const.}$$

It follows that each row of $\tilde{\Phi}^{-1}$ has the form $(q'' - bq - q'q)$ for some particular solution of (10.20). More precisely, since $\tilde{\Phi}$ is given by (10.18), we have

$$\tilde{\Phi}^{-1} = \begin{pmatrix} q_1'' - bq_1 - q_1'q_1 & q_1 \\ q_2'' - bq_2 - q_2'q_2 & q_2 \\ q_3'' - bq_3 - q_3'q_3 & q_3 \end{pmatrix}, \tag{10.22}$$

where

$$\begin{aligned} [p_0, q_1] &= 1, & [p_1, q_1] &= 0, & [p_4, q_1] &= 0, \\ [p_0, q_2] &= 0, & [p_1, q_2] &= 1, & [p_4, q_2] &= 0, \\ [p_0, q_3] &= 0, & [p_1, q_3] &= 0, & [p_4, q_3] &= 0. \end{aligned} \tag{10.23}$$

Then if $q_0 = q_2 + q_3$ we have

$$[p_0, q_0] = 0, \quad [p_1, q_0] = 1, \quad [p_4, q_0] = 1, \tag{10.24}$$

and from (10.18), (10.19), and (10.22) it follows that

$$K^{cusp}(x, y; b) = \frac{p_0(x)q_0(y) - p_0'(x)q_0'(y) + p_0''(x)q_0(y) - bp_0(x)q_0(y)}{2\pi i(x - y)}. \tag{10.25}$$

Recall that (10.21) has solutions with integral representations

$$p(z) = \int_{\Gamma} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2 + isz} ds, \tag{10.26}$$

where Γ is a contour in the complex plane that starts and ends at infinity at one of the angles $0, \pm\pi/2, \text{ or } \pi$. Similarly, there are solutions of (10.20) with integral representation

$$q(z) = \frac{1}{2\pi i} \int_{\Sigma} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 + itz} dt, \tag{10.27}$$

where Σ is a contour in the complex plane that starts and ends at infinity at one of the angles $\pm\pi/4$ or $\pm 3\pi/4$.

Lemma 10.2. *Let p and q be given by (10.26) and (10.27) such that $\Gamma \cap \Sigma = \emptyset$. Then*

$$[p, q] = 0.$$

If $\Gamma \cap \Sigma = \{z_0\}$ and Γ and Σ intersect transversally at z_0 , and if the contours are oriented so that Γ meets Σ in z_0 on the --side of Σ , then

$$[p, q] = 1.$$

Proof. We write $[p, q] = pq'' - p'q' + p''q - bpq$ as a double integral, and for convenience we take $z = 0$. So from (10.26) and (10.27),

$$\begin{aligned} [p, q] &= \frac{1}{2\pi i} \int_{\Sigma} \int_{\Gamma} (-t^2 - st - s^2 - b) e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 - \frac{1}{4}s^4 - \frac{b}{2}s^2} ds dt \\ &= \frac{1}{2\pi i} \int_{\Sigma} \int_{\Gamma} \frac{t^3 + bt - s^3 - bs}{s - t} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 - \frac{1}{4}s^4 - \frac{b}{2}s^2} ds dt. \end{aligned}$$

If $\Gamma \cap \Sigma = \emptyset$ then we can write this as

$$\begin{aligned} [p, q] &= \frac{1}{2\pi i} \int_{\Sigma} \int_{\Gamma} \frac{1}{s - t} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2} \frac{\partial}{\partial s} \left[e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2} \right] ds dt \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} \int_{\Sigma} \frac{1}{s - t} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2} \frac{\partial}{\partial t} \left[e^{\frac{1}{4}t^4 + \frac{b}{2}t^2} \right] dt ds \end{aligned}$$

and we can apply integration by parts to both inner integrals. The integrated terms vanish because of the choice of contours and the result is

$$\begin{aligned} [p, q] &= -\frac{1}{2\pi i} \int_{\Sigma} \int_{\Gamma} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 - \frac{1}{4}s^4 - \frac{b}{2}s^2} \frac{\partial}{\partial s} \left[\frac{1}{s - t} \right] ds dt \\ &\quad - \frac{1}{2\pi i} \int_{\Gamma} \int_{\Sigma} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 - \frac{1}{4}s^4 - \frac{b}{2}s^2} \frac{\partial}{\partial t} \left[\frac{1}{s - t} \right] dt ds = 0. \end{aligned}$$

If $\Gamma \cap \Sigma \neq \emptyset$ then we cannot make the splitting of integrals as above, and we have to proceed differently. If Γ and Σ intersect at z_0 as in the statement of the second part of the lemma, then we can deform contours so that Γ and Σ intersect in 0, and that for some $\delta > 0$, Σ contains the real interval $[-\delta, \delta]$ oriented from left to right, and Γ contains the vertical interval $[-i\delta, i\delta]$ oriented from bottom to top. Let $\varepsilon \in (0, \delta)$ and write $\Sigma_{\varepsilon} = \Sigma \setminus (-\varepsilon, \varepsilon)$. Then it follows as above that

$$\begin{aligned} [p, q] &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Sigma_{\varepsilon}} \int_{\Gamma} (-t^2 + st - s^2 - b) e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 - \frac{1}{4}s^4 - \frac{b}{2}s^2} ds dt \\ &= \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{2\pi i} \int_{\Sigma_{\varepsilon}} \int_{\Gamma} \frac{1}{s - t} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2} \frac{\partial}{\partial s} \left[e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2} \right] ds dt \right. \\ &\quad \left. + \frac{1}{2\pi i} \int_{\Gamma} \int_{\Sigma_{\varepsilon}} \frac{1}{s - t} e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2} \frac{\partial}{\partial t} \left[e^{\frac{1}{4}t^4 + \frac{b}{2}t^2} \right] dt ds \right]. \end{aligned}$$

If we now do an integration by parts, integrated terms at $\pm\varepsilon$ appear from the second double integral. The other terms vanish and the result is

$$[p, q] = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma} \left[\frac{1}{s + \varepsilon} - \frac{1}{s - \varepsilon} \right] e^{-\frac{1}{4}s^4 - \frac{b}{2}s^2} e^{\frac{1}{4}\varepsilon^4 + \frac{b}{2}\varepsilon^2} ds.$$

Now we deform Γ so that instead of the vertical segment $[-i\delta, i\delta]$ it contains the semi-circle $|s| = \delta, \operatorname{Re} s > 0$. Then we pick up a residue contribution from $s = \varepsilon$ which is equal to 1. The remaining integral vanishes in the limit $\varepsilon \rightarrow 0$, so that we find $[p, q] = 1$, as claimed by the lemma. \square

Lemma 10.2 allows us to compute $\tilde{\Phi}^{-1}$ explicitly. We claim that for $j = 1, 2, 3$,

$$q_j(z) = \frac{1}{2\pi i} \int_{\Sigma_j} e^{\frac{1}{4}t^4 + \frac{b}{2}t^2 + itz} dt, \tag{10.28}$$

where Σ_1 is a contour in the left half-plane from $e^{-3\pi i/4}\infty$ to $e^{3\pi i/4}\infty$, Σ_2 is a contour in the upper half-plane from $e^{\pi i/4}\infty$ to $e^{3\pi i/4}\infty$, and Σ_3 is a contour in the lower half-plane from $e^{-3\pi i/4}\infty$ to $e^{-\pi i/4}\infty$. Indeed, with these contours Σ_j , and taking note of the definition and orientation of Γ_0, Γ_1 , and Γ_4 in (8.11), we easily get from Lemma 10.2 that the relations (10.23) hold. Thus for $q_0 = q_2 + q_3$ we find that $q_0 = -iq$, where q is defined as in (1.8). Since $p_0 = 2\pi p$, it is then easy to check that the formula (10.25) for the kernel is equivalent to the formula (1.7) in the statement of the theorem. This completes the proof of Theorem 1.1.

Appendix A. Proof of Proposition 9.2

Let Σ_R be the contour depicted on Fig. 9, with orientation from the left to the right and in the positive direction on the circles. As usual, we will assume that the minus side of the contour is on the right.

By a simple arc on Σ_R we will mean a connected, relatively open, with respect to Σ_R , subset $\Sigma_R^0 \subset \Sigma_R$, which does not contain any triple point of Σ_R , a point where three curves meet. By $L^2(\Sigma_R)$ we will mean, as usual, the space of measurable functions with

$$\|f\|_2 = \left(\int_{\Sigma_R} |f|^2 |dz| \right)^{\frac{1}{2}} < \infty. \tag{A.1}$$

We have the following general proposition.

Proposition A.1. *Suppose that a 3×3 matrix-valued function $v(z), z \in \Sigma_R$, belongs to $L^2(\Sigma_R)$ and it is Lipschitz on some simple arc $\Sigma_R^0 \subset \Sigma_R$. Suppose also that on $\Sigma_R^0, v(z)$ solves the equation*

$$v(z) = I - \frac{1}{2\pi i} \int_{\Sigma_R} \frac{v(s)j_R^0(s)}{z_- - s} ds, \quad z \in \Sigma_R^0, \tag{A.2}$$

where z_- means the value of the limit of the integral from the minus side, and $j_R = I + j_R^0$. Then

$$R(z) = I - \frac{1}{2\pi i} \int_{\Sigma_R} \frac{v(s)j_R^0(s)}{z - s} ds, \quad z \in \mathbb{C} \setminus \Sigma_R, \tag{A.3}$$

satisfies on Σ_R^0 the jump condition,

$$R_+(z) = R_-(z)j_R(z), \quad z \in \Sigma_R^0. \tag{A.4}$$

Proof. From (A.2), (A.3),

$$R_-(z) = v(z), \quad z \in \Sigma_R. \tag{A.5}$$

By the jump property of the Cauchy transform,

$$R_+(z) - R_-(z) = v(z)j_R^0(z) = R_-(z)j_R^0(z), \tag{A.6}$$

hence $R_+(z) = R_-(z)j_R(z)$. Proposition A.1 is proved. \square

We will solve Eq. (A.2) by the series,

$$v(z) = v_0(z) + v_1(z) + v_2(z) + \dots, \tag{A.7}$$

where

$$v_0(z) = I; \quad v_j(z) = -\frac{1}{2\pi i} \int_{\Sigma_R} \frac{v_{j-1}(s)j_R^0(s)}{z_- - s} ds, \quad z \in \Sigma_R, \quad j \geq 1. \tag{A.8}$$

We will inductively estimate $v_j(z)$. We begin with some general definitions and results.

Introduce the operators

$$C_\Gamma^\pm v(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{v(s)}{z_\pm - s} ds, \quad z \in \Gamma, \tag{A.9}$$

where Γ is a contour on the complex plane. We assume that v is Lipschitz and L^2 integrable if Γ is unbounded. We have that

$$C_\Gamma^+ - C_\Gamma^- = \text{Id} \tag{A.10}$$

and

$$C_\Gamma^+ + C_\Gamma^- = C_\Gamma = -\frac{1}{\pi i} \text{v.p.} \int_\Gamma \frac{v(s)}{z - s} ds, \quad z \in \Gamma. \tag{A.11}$$

Suppose that the contour Γ is given by the parametric equations,

$$\Gamma = \{x = t, y = \varphi(t), -\infty < t < \infty\}, \tag{A.12}$$

where φ is uniformly Lipschitz, so that there exists $M \geq 0$ such that

$$|\varphi(x) - \varphi(y)| \leq M|x - y|. \tag{A.13}$$

Then as shown in [13], there exists an absolute constant K_0 such that

$$\|C_\Gamma f\|_2 \leq K_0(1 + M)^{10} \|f\|_2, \tag{A.14}$$

where

$$\|f\|_2 = \left(\int_\Gamma |f|^2 |dz| \right)^{\frac{1}{2}}. \tag{A.15}$$

This implies similar estimates for C_Γ^\pm . If $\Gamma_0 \subset \Gamma$ then

$$C_{\Gamma_0} = PC_\Gamma P, \quad Pf = \chi_{\Gamma_0} f, \tag{A.16}$$

hence

$$\|C_{\Gamma_0}\|_2 \leq \|C_\Gamma\|_2. \tag{A.17}$$

Therefore, estimate (A.14) holds, with the same constant, for any contour

$$\Gamma = \{x = t, y = \varphi(t), a < t < b\}. \tag{A.18}$$

Furthermore, it holds, with the same constant, for any complex linear transformation of contour (A.18). Let us denote by \mathcal{G}_M the set of all contours which can be obtained by a complex linear transformation from a contour (A.18), where φ satisfies (A.13) and is differentiable. Observe that any interval of a straight line belongs to \mathcal{G}_0 , and any circular arc of angular measure less than or equal to $\frac{\pi}{2}$ belongs to \mathcal{G}_1 .

Suppose now that $\Gamma = \Gamma_1 \cup \dots \cup \Gamma_m$ is a piecewise contour such that

- (1) Γ_j belongs to $\mathcal{G}_M, j = 1, \dots, m$;
- (2) the closed contours, $\overline{\Gamma_j}$ and $\overline{\Gamma_k}, j \neq k$, can intersect only at their end-points;
- (3) if $\overline{\Gamma_j}$ and $\overline{\Gamma_k}$ intersect then the angle between them at the intersection point is positive,

$$\angle(\Gamma_j, \Gamma_k) > \varepsilon > 0; \tag{A.19}$$

- (4) if Γ_j and $\Gamma_k, j \neq k$, are two infinite contour then they “well diverge” at infinity, so that there exists a constant $c > 0$ such that

$$|\gamma_j(s) - \gamma_k(t)| \geq c(|s| + |t|), \tag{A.20}$$

where γ_j, γ_k are the parametric equations of the contours Γ_j, Γ_k , induced by parametrization (A.12).

Theorem A.2. *If Γ is a piecewise contour which satisfies Conditions (1)–(4), then C_Γ is bounded in L^2 , and $\|C_\Gamma\|_2$ is estimated from above by a constant which depends only on the Lipschitz constants M_j of the contours $\Gamma_j, j = 1, \dots, m$, and on the constants ε and c of conditions (A.19), (A.20).*

Proof. We have to prove that for some $K_1 > 0$,

$$|(C_\Gamma f, g)| \leq K_1 \|f\|_2 \|g\|_2. \tag{A.21}$$

To that end, it is sufficient to prove that for some $K_2 > 0$,

$$|(C_\Gamma(\chi_{\Gamma_j} f), \chi_{\Gamma_k} g)| \leq K_2 \|f\|_2 \|g\|_2, \quad 1 \leq j, k \leq m. \tag{A.22}$$

For $j = k$, it follows from estimate (A.13) applied to a linear transformation of Γ_j . For $j \neq k$ it follows from (A.19), (A.20), and the estimate,

$$\frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{|f(s)g(t)| ds dt}{s+t} \leq \|f\|_2 \|g\|_2. \tag{A.23}$$

Theorem A.2 is proved. \square

When applied to the contour Σ_R , Theorem A.2 gives that there exists a constant K , independent of n , such that

$$\|C_{\Sigma_R}\|_2 \leq K. \tag{A.24}$$

By (A.10), (A.11) this implies that

$$\|C_{\Sigma_R}^{\pm}\|_2 \leq K_0 = \frac{K+1}{2}. \tag{A.25}$$

From (A.8) we have that

$$v_j = C_{\Sigma_R}^{-}(j_R^0 v_{j-1}), \quad j \geq 1. \tag{A.26}$$

Since

$$\|j_R^0 v_{j-1}\|_2 \leq \|j_R^0\|_C \|v_{j-1}\|_2 \tag{A.27}$$

and

$$\|j_R^0\|_C = \sup_{z \in \Sigma_R} |j_R^0(z)| \leq K_1 n^{-\frac{1}{6}}, \tag{A.28}$$

we obtain the recursive estimate,

$$\|v_j\|_2 \leq K n^{-\frac{1}{6}} \|v_{j-1}\|_2, \quad K = K_0 K_1. \tag{A.29}$$

For v_1 we have that

$$\|v_1\|_2 = \|C_{\Sigma_R}^{-}(j_R^0)\|_2 \leq K_0 \|j_R^0\|_2 \leq K_3 n^{-\frac{1}{6}-\frac{1}{8}}. \tag{A.30}$$

Thus,

$$\|v_j\|_2 \leq K_3 (K n^{-\frac{1}{6}})^j n^{-\frac{1}{8}}. \tag{A.31}$$

This implies the convergence of series (A.7) in L^2 , for large n . Let us discuss analytic properties of the functions v_j .

Denote

$$\Sigma_R = \bigcup_{l=1}^{16} \Sigma_R^l, \tag{A.32}$$

the partition of the contour Σ_R (see Fig. 9) into 16 simple arcs. Fix any $\varepsilon > 0$. Let z_0 be any point on Σ_R^l such that the distance from z_0 to the end-points of Σ_R^l is bigger than

$$\varepsilon_n = \varepsilon n^{-\frac{1}{4}}. \tag{A.33}$$

The function $j_R^0(z)$ can be analytically continued from Σ_R^l to the ε_n -neighborhood of the point z_0 ,

$$D(z_0, \varepsilon_n) = \{z : \text{dist}(z, z_0) < \varepsilon_n\}. \tag{A.34}$$

This implies that

$$v_1(z) = -\frac{1}{2\pi i} \int_{\Sigma_R} \frac{j_R^0(s)}{z-s} ds \tag{A.35}$$

can be also analytically continued from Σ_R^l to $D(z_0, \varepsilon_n)$, because we can deform the contour of integration, Σ_R . Then, inductively, we can analytically continue $v_j(z)$ from Σ_R^l to $D(z_0, \varepsilon_n)$, by deforming the contour of integration in (A.8). Observe that on the deformed contour we have the L^2 -estimate, (A.31), hence by the Cauchy-Schwarz inequality we obtain that

$$|v_j(z)| \leq K_4 \varepsilon^{-1} n^{\frac{1}{4}} (Kn^{-\frac{1}{6}})^{(j-1)} n^{-\frac{1}{8}}, \quad z \in D(z_0, \varepsilon_n/2). \tag{A.36}$$

This proves the convergence of series (A.7) in the neighborhood $D(z_0, \varepsilon_n/2)$ to an analytic $v(z)$. Thus, $v(z)$ is analytic on Σ_R outside of the triple points.

Observe that the function $R(z)$ defined by formula (A.3), denote it for a moment $\tilde{R}(z)$, coincides with $R(z)$ defined by (9.1). Indeed, both $\tilde{R}(z)$ and $R(z)$ solve the same RH problem, and if z_0 is any triple point of Σ_R , then in some neighborhood of z_0 ,

$$|\tilde{R}(z)| \leq C|z - z_0|^{-\frac{1}{2}}, \quad |R(z)| \leq C, \tag{A.37}$$

for some $C > 0$. For $\tilde{R}(z)$ it follows from (A.3) by the Cauchy-Schwarz inequality, and for $R(z)$ it is obvious from (9.1). If we consider now

$$X(z) = \tilde{R}(z)R(z)^{-1}, \tag{A.38}$$

then $X(z)$ has no jumps on Σ_R and in a neighborhood of the triple points it satisfies the estimate

$$|X(z)| \leq C|z - z_0|^{-\frac{1}{2}}. \tag{A.39}$$

Therefore, the triple points are removable singularities and $X(z)$ is analytic on \mathbb{C} . Also, $X(\infty) = I$, hence $X(z) = I$ everywhere on \mathbb{C} , and $\tilde{R}(z) = R(z)$. Now we can estimate $R(z)$.

From (A.3),

$$R(z) = I + \sum_{j=0}^{\infty} R_j(z), \quad R_j(z) = -\frac{1}{2\pi i} \int_{\Sigma_R} \frac{v_j(s)j_R^0(s)}{z - s} ds. \tag{A.40}$$

Suppose that

$$\text{dist}(z, \Sigma_R) > 0.1n^{-\frac{1}{4}}. \tag{A.41}$$

Then,

$$|R_0(z)| = \frac{1}{2\pi} \left| \int_{\Sigma_R} \frac{j_R^0(s)}{z - s} ds \right| \leq \frac{K_0 n^{-\frac{1}{6}}}{1 + |z|}, \tag{A.42}$$

and by (A.31),

$$|R_j(z)| \leq \frac{K_1 n^{\frac{1}{4}} (Kn^{-\frac{1}{6}})^{j+1} n^{-\frac{1}{8}}}{1 + |z|} = \frac{K_1 (Kn^{-\frac{1}{6}})^j n^{-\frac{1}{24}}}{1 + |z|}, \quad j \geq 1. \tag{A.43}$$

By summing all these inequalities from $j = 0$ to ∞ , we obtain that

$$R(z) = I + O\left(\frac{n^{-\frac{1}{6}}}{1 + |z|}\right), \quad \text{dist}(z, \Sigma_R) > 0.1n^{-\frac{1}{4}}. \tag{A.44}$$

In fact, the restriction $\text{dist}(z, \Sigma_R) > 0.1n^{-\frac{1}{4}}$ is not essential, because we can deform the contour Σ_R . This completes the proof of Proposition 9.2.

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