

Large n Limit of Gaussian Random Matrices with External Source, Part II*

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Abstract: We continue the study of the Hermitian random matrix ensemble with external source

$$\frac{1}{Z_n} e^{-n\text{Tr}(\frac{1}{2}M^2 - AM)} dM,$$

where A has two distinct eigenvalues $\pm a$ of equal multiplicity. This model exhibits a phase transition for the value $a = 1$, since the eigenvalues of M accumulate on two intervals for $a > 1$, and on one interval for $0 < a < 1$. The case $a > 1$ was treated in Part I, where it was proved that local eigenvalue correlations have the universal limiting behavior which is known for unitarily invariant random matrices, that is, limiting eigenvalue correlations are expressed in terms of the sine kernel in the bulk of the spectrum, and in terms of the Airy kernel at the edge. In this paper we establish the same results for the case $0 < a < 1$. As in Part I we apply the Deift/Zhou steepest descent analysis to a 3×3 -matrix Riemann-Hilbert problem. Due to the different structure of an underlying Riemann surface, the analysis includes an additional step involving a global opening of lenses, which is a new phenomenon in the steepest descent analysis of Riemann-Hilbert problems.

1. Introduction

This paper is a continuation of [6] to which we will frequently refer in this paper. It will be followed by a third part [8], which deals with the critical case. In these papers, we

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study the random matrix ensemble with external source A ,

$$\mu_n(dM) = \frac{1}{Z_n} e^{-n\text{Tr}(V(M)-AM)} dM, \quad (1.1)$$

defined on $n \times n$ Hermitian matrices M , with Gaussian potential

$$V(M) = \frac{1}{2} M^2, \quad (1.2)$$

and with external source

$$A = \text{diag}(\underbrace{a, \dots, a}_{n/2}, \underbrace{-a, \dots, -a}_{n/2}). \quad (1.3)$$

In the physics literature, the ensemble (1.1) was studied in a series of papers of Brézin and Hikami [9–13], and P. Zinn-Justin [33, 34]. The Gaussian ensemble, $V(M) = \frac{1}{2} M^2$, has been solved, in the large n limit, in the papers of Pastur [29] and Brézin-Hikami [9–13], by using spectral methods and a contour integration formula for the determinantal kernel. The contour integration technique has been extended in the recent work of Tracy and Widom [30] to the large n double scaling asymptotics of the determinantal kernel at the critical point. Our aim is to develop a completely different approach to the large n asymptotics of the Gaussian ensemble with external source. Our approach is based on the Riemann-Hilbert problem and it is applicable, in principle, to a general V . We develop the Deift-Zhou nonlinear steepest descent method for Riemann-Hilbert (RH) problems [18], thereby extending the works [3, 4, 16, 17, 25, 26] who treated the unitary invariant case (i.e., $A = 0$) with RH techniques.

While the unitary invariant case is connected with orthogonal polynomials [15, 27], the ensemble (1.1) is connected with multiple orthogonal polynomials [5]. These are characterized by a matrix RH problem [32], and the eigenvalue correlation kernel of (1.1) has a direct expression in terms of the solution of this RH problem, see [5, 14] and also formula (1.14) below. The RH problem for (1.1) has size $(r+1) \times (r+1)$ if r is the number of distinct eigenvalues of A . So with the choice (1.3), the RH problem is 3×3 -matrix valued.

The asymptotic analysis of RH problems has been mostly restricted to the 2×2 case. The analysis of larger size RH problems presents some novel technical features as already demonstrated in [6, 24]. In the present paper another new feature appears, namely at a critical stage in the analysis we perform a *global opening of lenses*. This global opening of lenses requires a global understanding of an associated Riemann surface, which is explicitly known for the Gaussian case (1.2). This is why we restrict ourselves to (1.2) although in principle our methods are applicable to a more general polynomial V .

The Gaussian case has some special relevance in its own right as well. Indeed, first of all we note that for (1.2) we can complete the square in (1.1), and then it follows that

$$M = M_0 + A, \quad (1.4)$$

where M_0 is a GUE matrix. So in the Gaussian case the ensemble (1.1) is an example of a random + deterministic model, see also [9–13].

A second interpretation of the Gaussian model comes from non-intersecting Brownian paths. This can be seen from the joint probability density for the eigenvalues

of M , which by the HarishChandra/Itzykson-Zuber formula [19, 27], takes the form

$$\frac{1}{Z_n} \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k) \det(e^{n\lambda_j a_k})_{j,k=1}^n \prod_{j=1}^n e^{-\frac{1}{2}n\lambda_j^2} \tag{1.5}$$

for the case (1.2). Here a_1, \dots, a_n are the eigenvalues of A , which are assumed to be all distinct in (1.5). In the case of coinciding eigenvalues of A we have to take the appropriate limit of (1.5), see formula (3.17) in [5].

Formula (1.5) also arises as the distribution of non-intersecting Brownian paths. Consider n independent Brownian motions (in fact Brownian bridges) on the line, starting at some fixed points $s_1 < s_2 < \dots < s_n$ at time $t = 0$, ending at some fixed points $b_1 < b_2 < \dots < b_n$ at time $t = 1$, and conditioned not to intersect for $t \in (0, 1)$. Then by a theorem of Karlin and McGregor [21], the joint probability density of the positions of the Brownian bridges at time $t \in (0, 1)$ is given by

$$p_n(x_1, \dots, x_n) = \frac{1}{C_n} \det(p(s_j, x_k; t))_{j,k=1}^n \det(p(x_j, b_k; 1-t))_{j,k=1}^n, \tag{1.6}$$

where $p(x, y; t)$ is the transition kernel of the Brownian motion and C_n is a normalization constant. Let us consider a scaled Brownian motion for which

$$p(x, y; t) = \sqrt{\frac{n}{2\pi t}} e^{-\frac{n(x-y)^2}{2t}}, \tag{1.7}$$

and let us take a limit when all initial points s_j converge to the origin. In this case formula (1.6) takes the form

$$p_n(x_1, \dots, x_n) = \frac{1}{\tilde{C}_n} \prod_{1 \leq j < k \leq n} (x_j - x_k) \det\left(e^{\frac{nx_j b_k}{1-t}}\right)_{j,k=1}^n \prod_{j=1}^n e^{-\frac{n}{2t(1-t)}x_j^2}. \tag{1.8}$$

This coincides with (1.5) if we make the identifications

$$\lambda_j = \frac{x_j}{\sqrt{t(1-t)}}, \quad a_k = b_k \sqrt{\frac{t}{1-t}}. \tag{1.9}$$

So at any time $t \in (0, 1)$ the positions of n non-intersecting Brownian bridges starting at 0 and ending at specified points are distributed as the eigenvalues of a Gaussian random matrix with external source.

The connection between random matrices and non-intersecting random paths is actually well-known, see e.g. the recent works [2, 20, 22, 28] and references cited therein.

P. Zinn-Justin showed that the m -point correlation functions for the eigenvalues of M have determinantal form

$$R_m(\lambda_1, \dots, \lambda_m) = \det(K_n(\lambda_j, \lambda_k))_{1 \leq j, k \leq m}. \tag{1.10}$$

It was shown in [5] that the average characteristic polynomial

$$P(z) = \mathbb{E}[\det(zI - M)]$$

is a multiple orthogonal polynomial of type II, which for the Gaussian case (1.2) is a multiple Hermite polynomial, see [1, 7], and that the correlation kernel K_n can be

expressed in terms of the solution of the RH problem for multiple orthogonal polynomials [32], see also [14].

We state the RH problem here for the Gaussian case (1.2) and for the external source (1.3) where n is even. Then the RH problem asks for a 3×3 matrix valued function Y satisfying the following:

- $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{3 \times 3}$ is analytic.
- For $x \in \mathbb{R}$, there is a jump

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.11}$$

where

$$w_1(x) = e^{-n(x^2/2-ax)}, \quad w_2(x) = e^{-n(x^2/2+ax)}, \tag{1.12}$$

and $Y_+(x)$ ($Y_-(x)$) denotes the limit of $Y(z)$ as $z \rightarrow x \in \mathbb{R}$ from the upper (lower) half-plane.

- As $z \rightarrow \infty$, we have

$$Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}. \tag{1.13}$$

The RH problem has a unique solution in terms of multiple Hermite polynomials and their Cauchy transforms [5, 32]. The correlation kernel K_n is expressed in terms of Y as follows:

$$K_n(x, y) = \frac{e^{-\frac{1}{4}n(x^2+y^2)}}{2\pi i(x-y)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0 \ e^{nay} \ e^{-nay}) Y^{-1}(y) Y(x). \tag{1.14}$$

Our goal is to analyze the above RH problem in the large n limit and to obtain from this scaling limits of the kernel (1.14) in various regimes. In this paper we consider the case $0 < a < 1$. The case $a > 1$ was considered in [6] and the critical case $a = 1$ will be considered in [8]. First we describe the limiting mean density of eigenvalues.

Theorem 1.1. *The limiting mean density of eigenvalues*

$$\rho(x) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) \tag{1.15}$$

exists for every $a > 0$. It satisfies

$$\rho(x) = \frac{1}{\pi} |\operatorname{Im} \xi(x)|, \tag{1.16}$$

where $\xi = \xi(x)$ is a solution of the cubic equation,

$$\xi^3 - x\xi^2 - (a^2 - 1)\xi + xa^2 = 0. \tag{1.17}$$

The support of ρ consists of those $x \in \mathbb{R}$ for which (1.17) has a non-real solution.

(a) For $0 < a < 1$, the support of ρ consists of one interval $[-z_1, z_1]$, and ρ is real analytic and positive on $(-z_1, z_1)$, and it vanishes like a square root at the edge points $\pm z_1$, i.e., there exists a constant $\rho_1 > 0$ such that

$$\rho(x) = \frac{\rho_1}{\pi} |x \mp z_1|^{1/2} (1 + o(1)) \quad \text{as } x \rightarrow \pm z_1, \quad x \in (-z_1, z_1). \quad (1.18)$$

(b) For $a = 1$, the support of ρ consists of one interval $[-z_1, z_1]$, and ρ is real analytic and positive on $(-z_1, 0) \cup (0, z_1)$, it vanishes like a square root at the edge points $\pm z_1$, and it vanishes like a third root at 0, i.e., there exists a constant $c > 0$ such that

$$\rho(x) = c|x|^{1/3} (1 + o(1)), \quad \text{as } x \rightarrow 0. \quad (1.19)$$

(c) For $a > 1$, the support of ρ consists of two disjoint intervals $[-z_1, -z_2] \cup [z_2, z_1]$ with $0 < z_2 < z_1$, ρ is real analytic and positive on $(-z_1, -z_2) \cup (z_2, z_1)$, and it vanishes like a square root at the edge points $\pm z_1, \pm z_2$.

Remark. Theorem 1.1 is a very special case of a theorem of Pastur [29] on the eigenvalues of a matrix $M = M_0 + A$, where M_0 is random and A is deterministic as in (1.4). Since in this paper our interest is in the case $0 < a < 1$, we show in Sect. 9 how Theorem 1.1 follows from our methods for this case. See [6] for the case $a > 1$.

Remark. Theorem 1.1 has the following interpretation in terms of non-intersecting Brownian motions starting at 0 and ending at some specified points b_j . We suppose n is even, and we let half of the b_j 's coincide with $b > 0$ and the other half with $-b$. Then as explained before, at time $t \in (0, 1)$ the (rescaled) positions of the Brownian paths coincide with the eigenvalues of the Gaussian random matrix with external source (1.3) where

$$a = b\sqrt{\frac{t}{1-t}}. \quad (1.20)$$

The phase transition at $a = 1$ corresponds to

$$t = t_c \equiv \frac{1}{1+b^2}.$$

So, by Theorem 1.1, the limiting distribution of the Brownian paths as $n \rightarrow \infty$ is supported by one interval when $t < t_c$ and by two intervals when $t > t_c$. At the critical time t_c the two groups of Brownian paths split, with one group ending at $t = 1$ at b and the other at $-b$.

As in [6] we formulate our main result in terms of a rescaled version of the kernel K_n ,

$$\hat{K}_n(x, y) = e^{n(h(x)-h(y))} K_n(x, y) \quad (1.21)$$

for some function h . The rescaling (1.21) does not affect the correlation functions (1.10).

Theorem 1.2. *Let $0 < a < 1$ and let z_1 and ρ be as in Theorem 1.1 (a). Then there is a function h such that the following hold for the rescaled kernel (1.21):*

(a) *For every $x_0 \in (-z_1, z_1)$ and $u, v \in \mathbb{R}$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho(x_0)} \hat{K}_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \quad (1.22)$$

(b) For every $u, v \in \mathbb{R}$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{(\rho_1 n)^{2/3}} \hat{K}_n \left(z_1 + \frac{u}{(\rho_1 n)^{2/3}}, z_1 + \frac{v}{(\rho_1 n)^{2/3}} \right) \\ = \frac{\text{Ai}(u) \text{Ai}'(v) - \text{Ai}'(u) \text{Ai}(v)}{u - v}, \end{aligned} \tag{1.23}$$

where Ai is the usual Airy function, and ρ_1 is the constant from (1.18).

Theorem 1.2 is similar to the main theorems Theorem 1.2 and Theorem 1.3 of [6]. It expresses that the local eigenvalue correlations show the universal behavior as $n \rightarrow \infty$, both in the bulk and at the edge, that is well-known from unitary random matrix models. So the result itself is not that surprising.

To obtain Theorem 1.2 we use the Deift/Zhou steepest descent method for RH problems and a main tool is the three-sheeted Riemann surface associated with Eq. (1.17) as in [6]. There is however an important technical difference with [6]. For $a > 1$, the branch points of the Riemann surface are all real, and they correspond to the four edge points $\pm z_1, \pm z_2$ of the support as described in Theorem 1.1 (c). For $a < 1$, two branch points are purely imaginary and they have no direct meaning for the problem at hand. The other two branch points are real and they correspond to the edge points $\pm z_1$ as in Theorem 1.1 (a). See Fig. 1 for the sheet structure of the Riemann surface. The branch points on the non-physical sheets result in a non-trivial modification of the steepest descent method. As already mentioned before, one of the steps involves a global opening of lenses, and this is the main new technical contribution of this paper.

The rest of the paper is devoted to the proof of the theorems with the Deift/Zhou steepest descent method for RH problems. It consists of a sequence of transformations which reduce the original RH problem to a RH problem which is normalized at infinity, and whose jump matrices are uniformly close to the identity as $n \rightarrow \infty$. In this paper there are four transformations $Y \mapsto U \mapsto T \mapsto S \mapsto R$. A main role is played by the Riemann surface (1.17) and certain λ -functions defined on it. These are introduced in the next section, and they are used in Sect. 3 to define the first transformation $Y \mapsto U$. This transformation has the effect of normalizing the RH problem at infinity, and in addition, of producing “good” jump matrices that are amenable to subsequent analysis. However, contrary to earlier works, some of the jump matrices for U have entries that are exponentially growing as $n \rightarrow \infty$. These exponentially growing entries disappear after the second transformation $U \mapsto T$ in Sect. 4 which involves the global opening of lenses.

The remaining transformation follows the pattern of [6, 16, 17] and other works. The transformation $T \mapsto S$ in Sect. 5 involves a local opening of lenses which turns the remaining oscillating entries into exponentially decaying ones. Then a parametrix for S is built in Sects. 6 and 7. In Sect. 6 a model RH problem is solved which provides the parametrix for S away from the branch points, and in Sect. 7 local parametrices are built around each of the branch points with the aid of Airy functions. Using this parametrix we define the final transformation $S \mapsto R$ in Sect. 8. It leads to a RH problem for R which is of the desired type: normalized at infinity and jump matrices tending to the identity as $n \rightarrow \infty$. Then R itself tends to the identity matrix as $n \rightarrow \infty$, which is then used in the final Sect. 9 to prove Theorems 1.1 and 1.2.

2. The Riemann Surface and λ -Functions

We start from the cubic equation (1.17) which we write now with the variable z instead of x

$$\xi^3 - z\xi^2 + (1 - a^2)\xi + za^2 = 0. \tag{2.1}$$

It defines a Riemann surface that will play a central role in the proof. The inverse mapping is given by the rational function

$$z = \frac{\xi^3 - (a^2 - 1)\xi}{\xi^2 - a^2}. \tag{2.2}$$

There are four branch points $\pm z_1, \pm iz_2$ with $z_1 > z_2 > 0$, which can be found as the images of the critical points under the inverse mapping. The mapping (2.2) has three inverses, $\xi_j(z), j = 1, 2, 3$, that behave near infinity as

$$\begin{aligned} \xi_1(z) &= z - \frac{1}{z} + O(1/z^2), \\ \xi_2(z) &= a + \frac{1}{2z} + O(1/z^2), \\ \xi_3(z) &= -a + \frac{1}{2z} + O(1/z^2). \end{aligned} \tag{2.3}$$

The sheet structure of the Riemann surface is determined by the way we choose the analytical continuations of the ξ_j 's.

It may be checked that ξ_1 has an analytic continuation to $\mathbb{C} \setminus [-z_1, z_1]$, which we take as the first sheet. The functions ξ_2 and ξ_3 have analytic continuations to $\mathbb{C} \setminus ([0, z_1] \cup [-iz_2, iz_2])$ and $\mathbb{C} \setminus ([-z_1, 0] \cup [-iz_2, iz_2])$, respectively, which we take to be the second and third sheets, respectively. So the second and third sheet are connected along $[-iz_2, iz_2]$, the first sheet is connected with the second sheet along $[0, z_1]$, and the first sheet is connected with the third sheet along $[-z_1, 0]$, see Fig. 1.

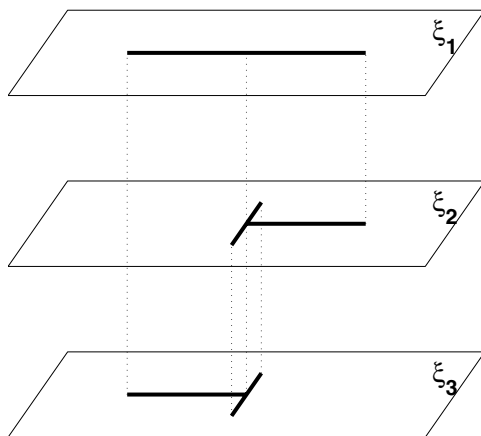


Fig. 1. The Riemann surface $\xi^3 - z\xi^2 + (1 - a^2)\xi + za^2 = 0$

We note the jump relations

$$\begin{aligned} \xi_{1\mp} &= \xi_{2\pm} && \text{on } (0, z_1), \\ \xi_{1\mp} &= \xi_{3\pm} && \text{on } (-z_1, 0), \\ \xi_{2\mp} &= \xi_{3\pm} && \text{on } (-iz_2, iz_2). \end{aligned} \tag{2.4}$$

The λ -functions are primitives of the ξ -functions $\lambda_j(z) = \int^z \xi_j(s)ds$, more precisely

$$\begin{aligned} \lambda_1(z) &= \int_{z_1}^z \xi_1(s)ds, \\ \lambda_2(z) &= \int_{z_1}^z \xi_2(s)ds, \\ \lambda_3(z) &= \int_{-z_{1+}}^z \xi_3(s)ds + \lambda_{1-}(-z_1). \end{aligned} \tag{2.5}$$

The path of integration for λ_3 lies in $\mathbb{C} \setminus ((-\infty, 0] \cup [-iz_2, iz_2])$, and it starts at the point $-z_1$ on the upper side of the cut. All three λ -functions are defined on their respective sheets of the Riemann surface with an additional cut along the negative real axis. Thus $\lambda_1, \lambda_2, \lambda_3$ are defined and analytic on $\mathbb{C} \setminus (-\infty, z_1], \mathbb{C} \setminus ((-\infty, z_1] \cup [-iz_2, iz_2])$, and $\mathbb{C} \setminus ((-\infty, 0] \cup [-iz_2, iz_2])$, respectively. Their behavior at infinity is

$$\begin{aligned} \lambda_1(z) &= \frac{1}{2}z^2 - \log z + \ell_1 + O(1/z), \\ \lambda_2(z) &= az + \frac{1}{2} \log z + \ell_2 + O(1/z), \\ \lambda_3(z) &= -az + \frac{1}{2} \log z + \ell_3 + O(1/z) \end{aligned} \tag{2.6}$$

for certain constants $\ell_j, j = 1, 2, 3$. The λ_j 's satisfy the following jump relations:

$$\begin{aligned} \lambda_{1\mp} &= \lambda_{2\pm} && \text{on } (0, z_1), \\ \lambda_{1-} &= \lambda_{3+} && \text{on } (-z_1, 0), \\ \lambda_{1+} &= \lambda_{3-} - \pi i && \text{on } (-z_1, 0), \\ \lambda_{2\mp} &= \lambda_{3\pm} && \text{on } (0, iz_2), \\ \lambda_{2\mp} &= \lambda_{3\pm} - \pi i && \text{on } (-iz_2, 0), \\ \lambda_{1+} &= \lambda_{1-} - 2\pi i && \text{on } (-\infty, -z_1), \\ \lambda_{2+} &= \lambda_{2-} + \pi i && \text{on } (-\infty, 0), \\ \lambda_{3+} &= \lambda_{3-} + \pi i && \text{on } (-\infty, -z_1), \end{aligned} \tag{2.7}$$

where the segment $(-iz_2, iz_2)$ is oriented upwards. We obtain (2.7) from (2.4), (2.5), and the values of the contour integrals around the cuts in the positive direction

$$\oint \xi_1(s)ds = -2\pi i, \quad \oint \xi_2(s)ds = \pi i, \quad \oint \xi_3(s)ds = \pi i,$$

which follow from (2.3).

Remark. We have chosen the segment $[-iz_2, iz_2]$ as the cut that connects the branch points $\pm iz_2$. We made this choice because of symmetry and ease of notation, but it is not essential. Instead we could have taken an arbitrary smooth curve lying in the region bounded by the four smooth curves in Fig. 2 (see the next section) that connect the points $x_0, iz_2, -x_0$, and $-iz_2$. For any such curve, the subsequent analysis would go through without any additional difficulty.

3. First Transformation $Y \mapsto U$

We define for $z \in \mathbb{C} \setminus (\mathbb{R} \cup [-iz_2, iz_2])$,

$$U(z) = \text{diag} \left(e^{-n\ell_1}, e^{-n\ell_2}, e^{-n\ell_3} \right) Y(z) \times \text{diag} \left(e^{n(\lambda_1(z) - \frac{1}{2}z^2)}, e^{n(\lambda_2(z) - az)}, e^{n(\lambda_3(z) + az)} \right). \tag{3.1}$$

This coincides with the first transformation in [6]. Then U solves the following RH problem:

- $U : \mathbb{C} \setminus (\mathbb{R} \cup [-iz_2, iz_2]) \rightarrow \mathbb{C}^{3 \times 3}$ is analytic.
- U satisfies the jumps

$$U_+ = U_- \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{1-})} & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & e^{n(\lambda_{2+} - \lambda_{2-})} & 0 \\ 0 & 0 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } \mathbb{R}, \tag{3.2}$$

and

$$U_+ = U_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{n(\lambda_{2+} - \lambda_{2-})} & 0 \\ 0 & 0 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } [-iz_2, iz_2]. \tag{3.3}$$

- $U(z) = I + O(1/z)$ as $z \rightarrow \infty$.

The asymptotic condition follows from (1.13), (2.6) and the definition of U .

The jump on the real line (3.2) takes on a different form on the four intervals $(-\infty, -z_1]$, $[-z_1, 0)$, $(0, z_1]$, and $[z_1, \infty)$. Indeed we get from (2.7), (3.2), and the fact that n is even,

$$U_+ = U_- \begin{pmatrix} 1 & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (-\infty, -z_1], \tag{3.4}$$

$$U_+ = U_- \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{1-})} & e^{n(\lambda_{2+} - \lambda_{1-})} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } (-z_1, 0), \tag{3.5}$$

$$U_+ = U_- \begin{pmatrix} e^{n(\lambda_{1+} - \lambda_{1-})} & 1 & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & e^{n(\lambda_{2+} - \lambda_{2-})} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (0, z_1), \tag{3.6}$$

$$U_+ = U_- \begin{pmatrix} 1 & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1, \infty). \tag{3.7}$$

Now to see what has happened it is important to know the sign of $\text{Re}(\lambda_j - \lambda_k)$ for $j \neq k$. Figure 2 shows the curves where $\text{Re} \lambda_j = \text{Re} \lambda_k$.

From each of the branch points $\pm z_1, \pm iz_2$ there are three curves emanating at equal angle of $2\pi/3$. We have $\text{Re} \lambda_1 = \text{Re} \lambda_2$ on the interval $[0, z_1]$ and on two unbounded curves from z_1 . Similarly, $\text{Re} \lambda_1 = \text{Re} \lambda_3$ on the interval $[-z_1, 0]$ and on two unbounded curves from $-z_1$. We have $\text{Re} \lambda_2 = \text{Re} \lambda_3$ on the curves that emanate from $\pm iz_2$. That is, on the vertical half-lines $[iz_2, +i\infty)$ and $(-i\infty, -iz_2]$ and on four other curves, before

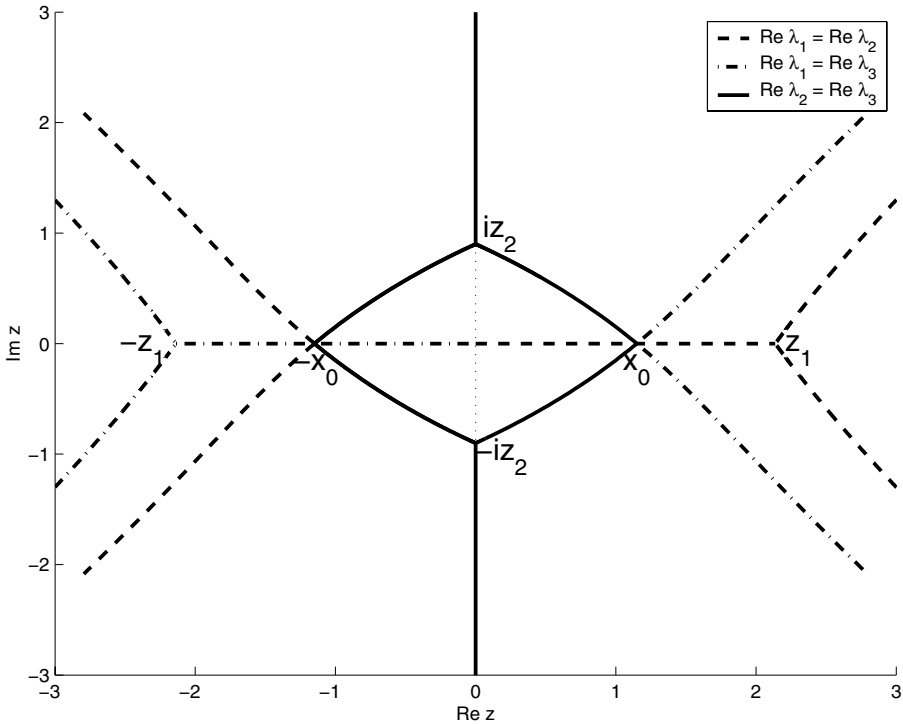


Fig. 2. Curves where $\text{Re } \lambda_1 = \text{Re } \lambda_2$ (dashed lines), $\text{Re } \lambda_1 = \text{Re } \lambda_3$ (dashed-dotted lines), and $\text{Re } \lambda_2 = \text{Re } \lambda_3$ (solid lines). This particular figure is for the value $a = 0.4$

they intersect the real axis. The points where they intersect the real axis are $\pm x_0$ for some $x_0 \in (0, z_1)$. After that point we have $\text{Re } \lambda_1 = \text{Re } \lambda_3$ for the curves in the right half-plane and $\text{Re } \lambda_1 = \text{Re } \lambda_2$ for the curves in the left half-plane. Figure 2 was produced with Matlab for the value $a = 0.4$. The picture is similar for other values of $a \in (0, 1)$. As $a \rightarrow 0+$ or $a \rightarrow 1-$, the imaginary branch points $\pm iz_2$ tend to the origin.

Using Fig. 2 and the asymptotic behavior (2.6) we can determine the ordering of $\text{Re } \lambda_j$, $j = 1, 2, 3$ in every domain in the plane. Indeed, in the domain on the right, bounded by the two unbounded curves emanating from z_1 , we have $\text{Re } \lambda_1 > \text{Re } \lambda_2 > \text{Re } \lambda_3$ because of (2.6). Then if we go to a neighboring domain, we pass a curve where $\text{Re } \lambda_1 = \text{Re } \lambda_2$, and so the ordering changes to $\text{Re } \lambda_2 > \text{Re } \lambda_1 > \text{Re } \lambda_3$. Continuing in this way, and also taking into account the cuts that we have for the λ_j 's, we find the ordering in any domain.

Inspecting the jump matrices for U in (3.3)–(3.7), we then find the following:

- (a) The non-zero off-diagonal entries in the jump matrices in (3.4) and (3.7) are exponentially small, and the jump matrices tend to the identity matrix as $n \rightarrow \infty$.
- (b) The non-constant diagonal entries in the jump matrices in (3.5) and (3.6) have modulus one, and they are rapidly oscillating for large n .
- (c) The (1, 2)-entry in the jump matrix in (3.5) is exponentially decreasing on $(-z_1, -x_0)$, but exponentially increasing on $(-x_0, 0)$ as $n \rightarrow \infty$. Similarly, the (1, 3)-entry in the jump matrix in (3.6) is exponentially decreasing on (x_0, z_1) , and exponentially increasing on $(0, x_0)$.

- (d) The entries in the jump matrix in (3.3) are real. The $(2, 2)$ -entry is exponentially increasing as $n \rightarrow \infty$, and the $(3, 3)$ -entry is exponentially decreasing.

The exponentially increasing entries observed in items (c) and (d) are undesirable, and this might lead to the impression that the first transformation $Y \mapsto U$ was not the right thing to do. However, after the second transformation which we do in the next section, all exponentially increasing entries miraculously disappear.

4. Second Transformation $U \mapsto T$

The second transformation involves the global opening of lenses already mentioned in the introduction. It is needed to turn the exponentially increasing entries in the jump matrices into exponentially decreasing ones.

Let Σ be a closed curve, consisting of a part in the left half-plane from $-iz_2$ to iz_2 , symmetric with respect to the real axis, plus its mirror image in the right half-plane. The part in the left half-plane lies entirely in the region where $\text{Re } \lambda_2 < \text{Re } \lambda_3$ and it intersects the negative real axis in a point $-x^*$ with $x^* > z_1$, see Fig. 3. So Σ avoids the region bounded by the curves from $\pm iz_2$ to $\pm x_0$. In a neighborhood of iz_2 we take Σ to be the analytic continuation of the curves where $\text{Re } \lambda_2 = \text{Re } \lambda_3$. As a result, this means that

$$\lambda_2 - \lambda_3 \text{ is real on } \Sigma \text{ in a neighborhood of } iz_2. \tag{4.1}$$

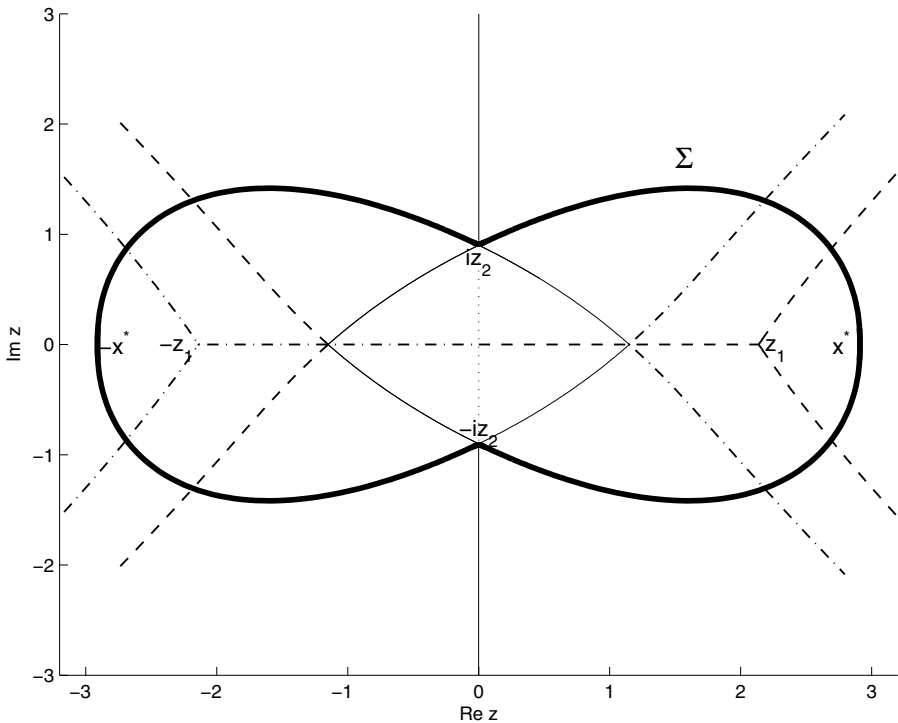


Fig. 3. Contour Σ which is such that $\text{Re } \lambda_2 < \text{Re } \lambda_3$ on the part of Σ in the left half-plane and $\text{Re } \lambda_2 > \text{Re } \lambda_3$ on the part of Σ in the right half-plane

This will be convenient for the construction of the local parametrix in Sect. 7.

The contour Σ encloses a bounded domain and we make the second transformation in that domain only. So we put $T = U$ outside Σ and inside Σ we put

$$\begin{aligned}
 T &= U \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{n(\lambda_2-\lambda_3)} & 1 \end{pmatrix} && \text{for } \operatorname{Re} z < 0 \text{ inside } \Sigma, \\
 T &= U \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_3-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} && \text{for } \operatorname{Re} z > 0 \text{ inside } \Sigma.
 \end{aligned}
 \tag{4.2}$$

Then T is defined and analytic outside the contours shown in Fig. 4. Using the jumps for U and the definition (4.2) we calculate the jumps for T on any part of the contour. We get different expressions for six real intervals, for the vertical segment $[-iz_2, iz_2]$, and for Σ (oriented clockwise) in the left and right half-planes. The result is that T satisfies the following RH problem:

- $T : \mathbb{C} \setminus (\mathbb{R} \cup [-iz_2, iz_2] \cup \Sigma) \rightarrow \mathbb{C}^{3 \times 3}$ is analytic.
- T satisfies the following jump relations on the real line

$$T_+ = T_- \begin{pmatrix} 1 & e^{n(\lambda_{2+}-\lambda_{1-})} & e^{n(\lambda_{3+}-\lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (-\infty, -x^*], \tag{4.3}$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & e^{n(\lambda_{3+}-\lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (-x^*, -z_1], \tag{4.4}$$

$$T_+ = T_- \begin{pmatrix} e^{n(\lambda_{1+}-\lambda_{1-})} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_{3+}-\lambda_{3-})} \end{pmatrix} \quad \text{on } (-z_1, 0), \tag{4.5}$$

$$T_+ = T_- \begin{pmatrix} e^{n(\lambda_{1+}-\lambda_{1-})} & 1 & 0 \\ 0 & e^{n(\lambda_{2+}-\lambda_{2-})} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (0, z_1), \tag{4.6}$$

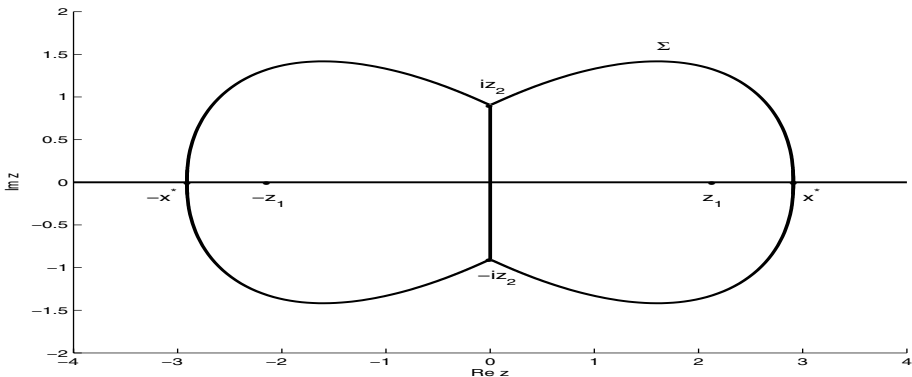


Fig. 4. T has jumps on the real line, the interval $[-iz_2, iz_2]$ and on Σ

$$T_+ = T_- \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1, x^*), \tag{4.7}$$

$$T_+ = T_- \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & e^{n(\lambda_3 - \lambda_1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1, \infty). \tag{4.8}$$

The jump on the vertical segment is

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_3 + -\lambda_3 -)} \end{pmatrix} \quad \text{on } [-iz_2, iz_2]. \tag{4.9}$$

The jumps on Σ are

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2 - \lambda_3)} & 1 \end{pmatrix} \quad \text{on } \{z \in \Sigma \mid \operatorname{Re} z < 0\}, \tag{4.10}$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \{z \in \Sigma \mid \operatorname{Re} z > 0\}. \tag{4.11}$$

- $T(z) = I + O(1/z)$ as $z \rightarrow \infty$.

Now the jump matrices are nice. Because of our choice of Σ we have that the jump matrices in (4.10) and (4.11) converge to the identity matrix as $n \rightarrow \infty$. Also the jump matrices in (4.3), (4.4), (4.7) and (4.8) converge to the identity matrix as $n \rightarrow \infty$. The (3, 3)-entry in the jump matrix in (4.9) is exponentially small, so that this matrix tends

to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

The jump matrices in (4.6) and (4.7) have oscillatory entries on the diagonal, and they are turned into exponential decaying off-diagonal entries by opening a (local) lens around $(-z_1, z_1)$. This is the next transformation.

5. Third Transformation $T \mapsto S$

We are now going to open up a lens around (z_1, z_1) as in Fig. 5. There is no need to treat 0 as a special point.

The jump matrix on $(-z_1, 0)$, see (4.5), has factorization

$$\begin{aligned} T_-^{-1} T_+ &= \begin{pmatrix} e^{n(\lambda_1 - \lambda_3)_+} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_1 - \lambda_3)_-} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1 - \lambda_3)_-} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1 - \lambda_3)_+} & 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.1}$$

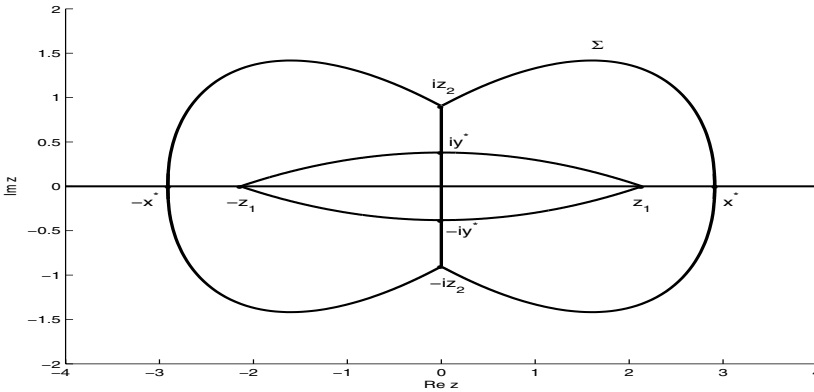


Fig. 5. Opening of lens around $[-z_1, z_1]$. The new matrix-valued function S has jumps on the real line, the interval $[-iz_2, iz_2]$, on Σ , and on the upper and lower lips of the lens around $[-z_1, z_1]$

and the jump matrix on $(0, z_1)$, see (4.6), has factorization

$$\begin{aligned}
 T_-^{-1}T_+ &= \begin{pmatrix} e^{n(\lambda_1-\lambda_2)_+} & 1 & 0 \\ 0 & e^{n(\lambda_1-\lambda_2)_-} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)_-} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)_+} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2}
 \end{aligned}$$

We open up the lens on $[-z_1, z_1]$ and we make sure that it stays inside Σ . We assume that the lens is symmetric with respect to the real and imaginary axis. The point where the upper lip intersects the imaginary axis is called iy^* . Then we define $S = T$ outside the lens and

$$\begin{aligned}
 S &= T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{n(\lambda_1-\lambda_3)} & 0 & 1 \end{pmatrix} && \text{in upper part of the lens in left half-plane,} \\
 S &= T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1-\lambda_3)} & 0 & 1 \end{pmatrix} && \text{in lower part of the lens in left half-plane,} \\
 S &= T \begin{pmatrix} 1 & 0 & 0 \\ -e^{n(\lambda_1-\lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{in upper part of the lens in right half-plane,} \\
 S &= T \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1-\lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{in lower part of the lens in right half-plane.}
 \end{aligned} \tag{5.3}$$

Outside the lens, the jumps for S are as those for T , while on $[-z_1, z_1]$ and on the upper and lower lips of the lens, the jumps are according to the factorizations (5.1) and (5.2). The result is that S satisfies the following RH problem:

- S is analytic outside the real line, the vertical segment $[-iz_2, iz_2]$, the curve Σ , and the upper and lower lips of the lens around $[-z_1, z_1]$.
- S satisfies the following jumps on the real line:

$$S_+ = S_- \begin{pmatrix} 1 & e^{n(\lambda_{2+} - \lambda_{1-})} & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (-\infty, -x^*], \quad (5.4)$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & e^{n(\lambda_{3+} - \lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (-x^*, -z_1], \quad (5.5)$$

$$S_+ = S_- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{on } (-z_1, 0), \quad (5.6)$$

$$S_+ = S_- \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (0, z_1), \quad (5.7)$$

$$S_+ = S_- \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1, x^*), \quad (5.8)$$

$$S_+ = S_- \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & e^{n(\lambda_3 - \lambda_1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [x^*, \infty). \quad (5.9)$$

S has the following jumps on the segment $[-iz_2, iz_2]$:

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } (-iz_2, -iy^*), \quad (5.10)$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ e^{n(\lambda_{1-} - \lambda_{3-})} & -1 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } (-iy^*, 0), \quad (5.11)$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -e^{n(\lambda_{1-} - \lambda_{3-})} & -1 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } (0, iy^*), \quad (5.12)$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} \quad \text{on } (iy^*, iz_2). \quad (5.13)$$

The jumps on Σ are

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2 - \lambda_3)} & 1 \end{pmatrix} \quad \text{on } \{z \in \Sigma \mid \operatorname{Re} z < 0\}, \quad (5.14)$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \{z \in \Sigma \mid \operatorname{Re} z > 0\}. \quad (5.15)$$

Finally, on the upper and lower lips of the lens, we find jumps

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1 - \lambda_3)} & 0 & 1 \end{pmatrix} \quad \text{on the lips of the lens in the left half-plane.} \tag{5.16}$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the lips of the lens in the right half-plane.} \tag{5.17}$$

- $S(z) = I + O(1/z)$ as $z \rightarrow \infty$.

So now we have 14 different jump matrices (5.4)–(5.17). As $n \rightarrow \infty$, all these jumps have limits. Most of the limits are the identity matrix, except for the jumps on $(-z_1, z_1)$, see (5.6) and (5.7), and on $(-iz_2, iz_2)$, see (5.10)–(5.13). In the next section we will solve explicitly the limiting model RH problem. The solution to the model problem will be further used in the construction of parametrix away from the branch points.

6. Parametrix Away from Branch Points

The model RH problem is the following. Find N such that

- $N : \mathbb{C} \setminus ([-z_1, z_1] \cup [-iz_2, iz_2]) \rightarrow \mathbb{C}^{3 \times 3}$ is analytic.
- N satisfies the jumps

$$N_+ = N_- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{on } [-z_1, 0), \tag{6.1}$$

$$N_+ = N_- \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (0, z_1], \tag{6.2}$$

$$N_+ = N_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{on } [-iz_2, iz_2]. \tag{6.3}$$

- $N(z) = I + O(1/z)$ as $z \rightarrow \infty$.

To solve the model RH problem we lift it to the Riemann surface (2.1) with the sheet structure as in Fig. 1, see also [6, 24], where the same technique was used. Consider to that end the range of the functions ξ_k on the complex plane, $\Omega_k = \xi_k(\mathbb{C})$ for $k = 1, 2, 3$. Then $\Omega_1, \Omega_2, \Omega_3$ give a partition of the complex plane into three regions, see Fig. 6. In this figure q, p and p_0 are such that

$$\begin{aligned} q &= \xi_1(z_1) = \xi_2(z_1) = -\xi_1(-z_1) = -\xi_3(-z_1), \\ ip &= -\xi_2(iz_2) = -\xi_3(iz_2) = \xi_2(-iz_2) = \xi_3(-iz_2), \\ ip_0 &= \xi_{1+}(0) = -\xi_{1-}(0). \end{aligned} \tag{6.4}$$

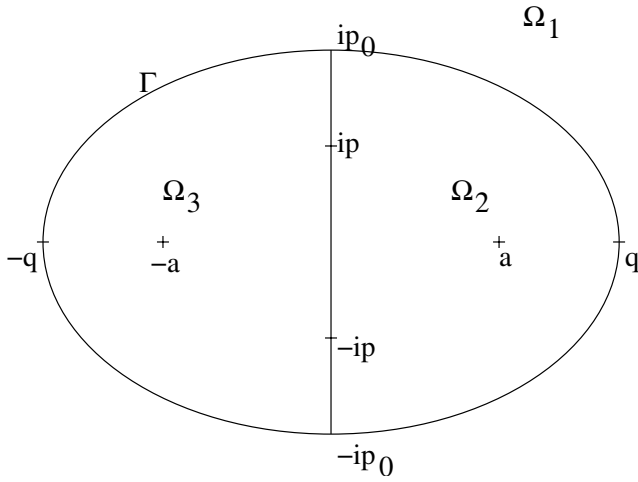


Fig. 6. Partition of the complex ξ -plane

Let Γ be the boundary of Ω_1 . Then we have

$$\begin{aligned} \xi_{1\pm}([-z_1, z_1]) &= \Gamma \cap \{\pm \text{Im } z \geq 0\}, \\ \xi_{2-}([-iz_2, 0]) &= [ip, ip_0], \quad \xi_{2-}([0, iz_2]) = [-ip_0, -ip], \\ \xi_{3-}([-iz_2, 0]) &= [ip, 0], \quad \xi_{3-}([0, iz_2]) = [0, -ip], \end{aligned} \tag{6.5}$$

and $\xi_{2\pm}(iy) = \xi_{3\mp}(iy)$ for $-z_2 \leq y \leq z_2$. According to our agreement, on the interval $-iz_2 \leq y \leq iz_2$ the minus side is on the right.

We are looking for a solution N in the following form:

$$N(z) = \begin{pmatrix} N_1(\xi_1(z)) & N_1(\xi_2(z)) & N_1(\xi_3(z)) \\ N_2(\xi_1(z)) & N_2(\xi_2(z)) & N_2(\xi_3(z)) \\ N_3(\xi_1(z)) & N_3(\xi_2(z)) & N_3(\xi_3(z)) \end{pmatrix}, \tag{6.6}$$

where $N_1(\xi)$, $N_2(\xi)$, $N_3(\xi)$ are three scalar analytic functions on $\mathbb{C} \setminus (\Gamma \cup [-ip_0, ip_0])$. To satisfy the jump conditions on $N(z)$ we need the following jump relations for $N_j(\xi)$, $j = 1, 2, 3$:

$$\begin{aligned} N_{j+}(\xi) &= N_{j-}(\xi), \quad \xi \in (\Gamma \cap \{\text{Im } z \leq 0\}) \cup [-ip_0, -ip] \cup [ip, ip_0], \\ N_{j+}(\xi) &= -N_{j-}(\xi), \quad \xi \in (\Gamma \cap \{\text{Im } z \geq 0\}) \cup [-ip, ip]. \end{aligned} \tag{6.7}$$

So the N_j 's are actually analytic across the curve Γ in the lower half-plane and on the segments $[ip, ip_0]$ and $[-ip_0, -ip]$. What remains are the curve Γ in the upper half-plane and the segment $[-ip, ip]$, where the functions change sign. Since $\xi_1(\infty) = \infty$, $\xi_2(\infty) = a$, $\xi_3(\infty) = -a$, then to satisfy $N(\infty) = I$ we require

$$\begin{aligned} N_1(\infty) &= 1, & N_1(a) &= 0, & N_1(-a) &= 0; \\ N_2(\infty) &= 0, & N_2(a) &= 1, & N_2(-a) &= 0; \\ N_3(\infty) &= 0, & N_3(a) &= 0, & N_3(-a) &= 1. \end{aligned} \tag{6.8}$$

Thus, we obtain three scalar RH problems on N_1, N_2, N_3 . Equations (6.7)–(6.8) have the following solution:

$$N_1(\xi) = \frac{\xi^2 - a^2}{\sqrt{(\xi^2 + p^2)(\xi^2 - q^2)}}, \quad N_{2,3}(\xi) = c_{2,3} \frac{\xi \pm a}{\sqrt{(\xi^2 + p^2)(\xi^2 - q^2)}}, \quad (6.9)$$

with cuts at $\Gamma \cap \{\text{Im } \xi \geq 0\}$ and $[-ip, ip]$. The constants $c_{2,3}$ are determined by the equations $N_{2,3}(\pm a) = 1$. We have that

$$(\xi^2 + p^2)(\xi^2 - q^2) = \xi^4 - (1 + 2a^2)\xi^2 + (a^2 - 1)a^2 \equiv R(\xi; a), \quad (6.10)$$

and as in Sect. 6 of [6], we obtain $c_2 = c_3 = -\frac{i}{\sqrt{2}}$. Thus, the solution to the model RH problem is given by

$$N(z) = \begin{pmatrix} \frac{\xi_1^2(z) - a^2}{\sqrt{R(\xi_1(z); a)}} & \frac{\xi_2^2(z) - a^2}{\sqrt{R(\xi_2(z); a)}} & \frac{\xi_3^2(z) - a^2}{\sqrt{R(\xi_3(z); a)}} \\ -i \frac{\xi_1(z) + a}{\sqrt{2R(\xi_1(z); a)}} & -i \frac{\xi_2(z) + a}{\sqrt{2R(\xi_2(z); a)}} & -i \frac{\xi_3(z) + a}{\sqrt{2R(\xi_3(z); a)}} \\ -i \frac{\xi_1(z) - a}{\sqrt{2R(\xi_1(z); a)}} & -i \frac{\xi_2(z) - a}{\sqrt{2R(\xi_2(z); a)}} & -i \frac{\xi_3(z) - a}{\sqrt{2R(\xi_3(z); a)}} \end{pmatrix}, \quad (6.11)$$

with cuts on $[-z_1, z_1]$ and $[-iz_2, iz_2]$.

7. Local Parametrics

Near the branch points N will not be a good approximation to S . We need a local analysis near each of the branch points. In a small circle around each of the branch points, the parametrix P should have the same jumps as S , and on the boundary of the circle P should match with N in the sense that

$$P(z) = N(z) (I + O(1/n)) \quad (7.1)$$

uniformly for z on the boundary of the circle.

The construction of P near the real branch points $\pm z_1$ makes use of Airy functions and it is the same as the one given in [6, Sect. 7] for the case $a > 1$. The parametrix near the imaginary branch points $\pm iz_2$ is also constructed with Airy functions. We give the construction near iz_2 . We want an analytic P in a neighborhood of iz_2 with jumps

$$\begin{aligned} P_+ &= P_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2 - \lambda_3)} & 1 \end{pmatrix} && \text{on left contour,} \\ P_+ &= P_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} && \text{on right contour,} \\ P_+ &= P_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_{3+} - \lambda_{3-})} \end{pmatrix} && \text{on vertical part.} \end{aligned} \quad (7.2)$$

In addition we need the matching condition (7.1). Except for the matching condition (7.1), the problem is a 2×2 problem.

Let us consider $\lambda_2 - \lambda_3$ near the branch point iz_2 . We know that $(\lambda_2 - \lambda_3)(iz_2) = 0$, see (2.7) and since $\xi_2 - \xi_3$ has square root behavior at iz_2 it follows that

$$(\lambda_2 - \lambda_3)(z) = \int_{iz_2}^z (\xi_2(s) - \xi_3(s))ds = (z - iz_2)^{3/2}h(z)$$

with an analytic function h with $h(iz_2) \neq 0$. So we can take a 2/3-power and obtain a conformal map. To be precise, we note that

$$\arg((\lambda_2 - \lambda_3)(iy)) = \pi/2, \quad \text{for } y > z_2,$$

and so we define

$$f(z) = \left[\frac{3}{4}(\lambda_2 - \lambda_3)(z) \right]^{2/3} \tag{7.3}$$

such that

$$\arg f(z) = \pi/3, \quad \text{for } z = iy, y > z_2.$$

Then $s = f(z)$ is a conformal map, which maps $[0, iz_2]$ into the ray $\arg s = -\frac{2\pi}{3}$, and which maps the parts of Σ near iz_2 in the right and left half-planes into the rays $\arg s = 0$ and $\arg s = \frac{2\pi}{3}$, respectively. [Recall that $\lambda_2 - \lambda_3$ is real on these contours, see (4.1).]

We choose P of the form

$$P(z) = E(z)\Phi \left(n^{2/3} f(z) \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{1}{2}n(\lambda_2 - \lambda_3)} & 0 \\ 0 & 0 & e^{-\frac{1}{2}n(\lambda_2 - \lambda_3)} \end{pmatrix}, \tag{7.4}$$

where E is analytic. In order to satisfy the jump conditions (7.2) we want that Φ is defined and analytic in the complex s -plane cut along the three rays $\arg s = k\frac{2\pi}{3}$, $k = -1, 0, 1$, and there it has jumps

$$\begin{aligned} \Phi_+ &= \Phi_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} && \text{for } \arg s = 2\pi/3, \\ \Phi_+ &= \Phi_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} && \text{for } \arg s = -2\pi/3, \\ \Phi_+ &= \Phi_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} && \text{for } \arg s = 0. \end{aligned} \tag{7.5}$$

Put $y_0(s) = \text{Ai}(s)$, $y_1(s) = \omega \text{Ai}(\omega s)$, $y_2(s) = \omega^2 \text{Ai}(\omega^2 s)$ with $\omega = 2\pi/3$ and Ai the standard Airy function. Then we take Φ as

$$\begin{aligned} \Phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_0 & -y_2 \\ 0 & y'_0 & -y'_2 \end{pmatrix} && \text{for } 0 < \arg s < 2\pi/3, \\ \Phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_0 & y_1 \\ 0 & y'_0 & y'_1 \end{pmatrix} && \text{for } -2\pi/3 < \arg s < 0, \\ \Phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -y_1 & -y_2 \\ 0 & -y'_1 & -y'_2 \end{pmatrix} && \text{for } 2\pi/3 < \arg s < 4\pi/3. \end{aligned} \tag{7.6}$$

This Φ satisfies the jumps (7.5). In order to achieve the matching (7.1) we define the prefactor E as

$$E = NL^{-1} \tag{7.7}$$

with

$$L = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & n^{-1/6} f^{-1/4} & 0 \\ 0 & 0 & n^{1/6} f^{1/4} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -1 & i \end{pmatrix}, \tag{7.8}$$

where $f^{1/4}$ has a branch cut along the vertical segment $[0, iz_2]$ and it is real and positive where f is real and positive. The matching condition (7.1) now follows from the asymptotics of the Airy function and its derivative

$$\begin{aligned} \text{Ai}(s) &= \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{2}{3}s^{3/2}} \left(1 + O\left(s^{-3/2}\right) \right), \\ \text{Ai}'(s) &= -\frac{1}{2\sqrt{\pi}} s^{1/4} e^{-\frac{2}{3}s^{3/2}} \left(1 + O\left(s^{-3/2}\right) \right), \end{aligned}$$

as $s \rightarrow \infty, |\arg s| < \pi$. On the cut we have $f_+^{1/4} = if_-^{1/4}$. Then (7.8) gives

$$L_+ = L_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

which is the same jump as satisfied by N , see (6.3). This implies that $E = NL^{-1}$ is analytic in a punctured neighborhood of iz_2 . Since the entries of N and L have at most fourth-root singularities, the isolated singularity is removable, and E is analytic. It follows that P defined by (7.4) does indeed satisfy the jumps (7.2) and the matching condition (7.1).

A similar construction gives the parametrix in the neighborhood of $-iz_2$.

8. Fourth Transformation $S \mapsto R$

Having constructed N and P , we define the final transformation by

$$\begin{aligned} R(z) &= S(z)N(z)^{-1} && \text{away from the branch points,} \\ R(z) &= S(z)P(z)^{-1} && \text{near the branch points.} \end{aligned} \tag{8.1}$$

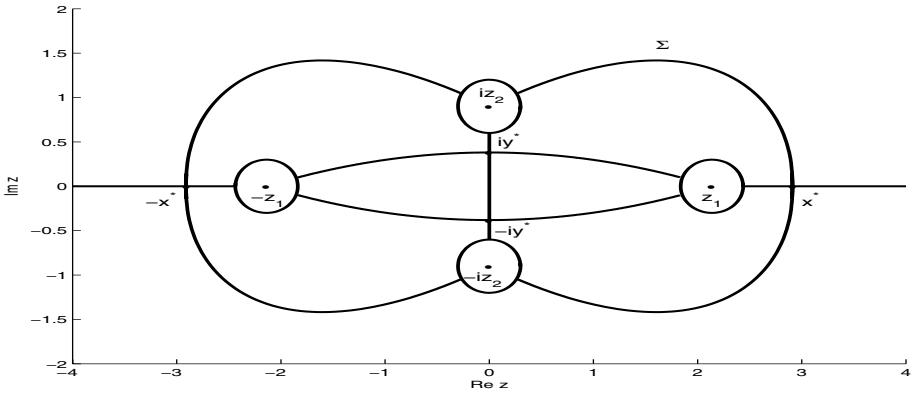


Fig. 7. R has jumps on this system of contours

Since jumps of S and N coincide on the interval $(-z_1, z_1)$ and the jumps of S and P coincide inside the disks around the branch points, we obtain that R is analytic outside a system of contours as shown in Fig. 7.

On the circles around the branch points there is a jump

$$R_+ = R_-(I + O(1/n)), \tag{8.2}$$

which follows from the matching condition (7.1). On the remaining contours, the jump is

$$R_+ = R_-(I + O(e^{-cn})) \tag{8.3}$$

for some $c > 0$. Since we also have the asymptotic condition $R(z) = I + O(1/z)$ as $z \rightarrow \infty$, we may conclude as in [5, Sect. 8] that

$$R(z) = I + O\left(\frac{1}{n(|z| + 1)}\right) \quad \text{as } n \rightarrow \infty, \tag{8.4}$$

uniformly for $z \in \mathbb{C}$, see also [15–17, 23].

9. Proof of Theorems 1.1 and 1.2

We follow the expression for the kernel K_n as we make the transformations $Y \mapsto U \mapsto T \mapsto S$. From (1.14) and the transformation (3.1) it follows that K_n has the following expression in terms of U , for any $x, y \in \mathbb{R}$,

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{n\lambda_{2+}(y)} & e^{n\lambda_{3+}(y)} \end{pmatrix} U_+^{-1}(y)U_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix}. \tag{9.1}$$

Then from (4.2) we obtain for $y \geq 0$ inside the contour Σ , and for any $x \in \mathbb{R}$,

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{n\lambda_{2+}(y)} & 0 \end{pmatrix} T_+^{-1}(y)T_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix}, \tag{9.2}$$

and from (5.3), we have when $x, y \in [0, z_1)$,

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & e^{n\lambda_{2+}(y)} & 0 \end{pmatrix} S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ e^{-n\lambda_{2+}(x)} \\ 0 \end{pmatrix}. \tag{9.3}$$

Since λ_{1+} and λ_{2+} are each others complex conjugates on $[0, z_1)$, we can rewrite (9.3) for $x, y \in [0, z_1)$ as

$$K_n(x, y) = \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{ni \operatorname{Im} \lambda_{1+}(y)} & e^{-ni \operatorname{Im} \lambda_{1+}(y)} & 0 \end{pmatrix} \times S_+^{-1}(y)S_+(x) \begin{pmatrix} e^{-ni \operatorname{Im} \lambda_{1+}(x)} \\ e^{ni \operatorname{Im} \lambda_{1+}(x)} \\ 0 \end{pmatrix}, \tag{9.4}$$

where

$$h(x) = \operatorname{Re} \lambda_{1+}(x) - \frac{1}{4}x^2. \tag{9.5}$$

Note that (9.4) is exactly the same as Eq. (5.14) in [6]. Therefore we can almost literally follow the proofs in Sect. 9 of [6] to complete the proof of Theorem 1.1 and 1.2.

Indeed as in [6] the limiting mean density (1.15) follows from (9.4) and (8.4) in case $x > 0$, where

$$\rho(x) = \frac{1}{\pi} \operatorname{Im} \xi_{1+}(x), \quad x \in \mathbb{R}. \tag{9.6}$$

The case $x < 0$ follows in the same way and also by symmetry. Recalling that the choice of the cut $[-iz_2, iz_2]$ was arbitrary as remarked at the end of Sect. 2, we note that we might as well have done the asymptotic analysis on a contour that does not pass through 0, so that we obtain (1.15) for $x = 0$ as well. The statement in part (a) on the behavior of ρ follows immediately from (9.6) and the properties of ξ_1 as the inverse mapping of (2.2).

This completes the proof of Theorem 1.1.

The proof of part (a) of Theorem 1.2 for the case $x_0 > 0$ follows from (9.4) and (8.4) exactly as in Sect. 9 of [6]. The case $x_0 < 0$ follows by symmetry, and the case $x_0 = 0$ follows as well, since we might have done the asymptotic analysis on a cut different from $[-iz_2, iz_2]$, as just noted above. The proof of part (b) follows as in [6] as well. Note however that the proof of part (b) relies on the local parametrix at the branch point z_1 , which we have not specified explicitly in Sect. 7. However, the formulas are the same as those in [6] and the proof can be copied. This completes the proof of Theorem 1.2.

References

1. Aptekarev, A.I., Branquinho, A., Van Assche, W.: Multiple orthogonal polynomials for classical weights. *Trans. Amer. Math. Soc.* **355**, 3887–3914 (2003)
2. Baik, J.: Random vicious walks and random matrices. *Commun. Pure Appl. Math.* **53**, 1385–1410 (2000)
3. Bleher, P., Its, A.: Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and the universality in the matrix model. *Ann. Math.* **150**, 185–266 (1999)
4. Bleher, P., Its, A.: Double scaling limit in the random matrix model. The Riemann-Hilbert approach. *Commun. Pure Appl. Math.* **56**, 433–516 (2003)

5. Bleher, P.M., Kuijlaars, A.B.J.: Random matrices with external source and multiple orthogonal polynomials. *Internat. Math. Research Notices* 2004, **3**, 109–129 (2004)
6. Bleher, P.M., Kuijlaars, A.B.J.: Large n limit of Gaussian random matrices with external source. Part I. *Commun. Math. Phys.* **252**, 43–76 (2004)
7. Bleher, P.M., Kuijlaars, A.B.J.: Integral representations for multiple Hermite and multiple Laguerre polynomials. <http://arxiv.org/abs/math.CA/0406616>, 2004 to appear in *Annales de l'Institut Fourier*
8. Bleher, P.M., Kuijlaars, A.B.J.: Large n limit of Gaussian random matrices with external source, Part III: double scaling limit in the critical case. In preparation
9. Brézin, E., Hikami, S.: Spectral form factor in a random matrix theory. *Phys. Rev. E* **55**, 4067–4083 (1997)
10. Brézin, E., Hikami, S.: Correlations of nearby levels induced by a random potential. *Nucl. Phys. B* **479**, 697–706 (1996)
11. Brézin, E., Hikami, S.: Extension of level spacing universality. *Phys. Rev. E* **56**, 264–269 (1997)
12. Brézin, E., Hikami, S.: Universal singularity at the closure of a gap in a random matrix theory. *Phys. Rev. E* **57**, 4140–4149 (1998)
13. Brézin, E., Hikami, S.: Level spacing of random matrices in an external source. *Phys. Rev. E* **58**, 7176–7185 (1998)
14. Daems, E., Kuijlaars, A.B.J.: A Christoffel-Darboux formula for multiple orthogonal polynomials. *J. Approx. Theory* **130**, 190–202 (2004)
15. Deift, P.: Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert approach. *Courant Lecture Notes in Mathematics*, Vol. **3**, Providence R.I.: Amer. Math. Soc., 1999
16. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Uniform asymptotics of polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Commun. Pure Appl. Math.* **52**, 1335–1425 (1999)
17. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Commun. Pure Appl. Math.* **52**, 1491–1552 (1999)
18. Deift, P., Zhou, X.: A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. *Ann. Math.* **137**, 295–368 (1993)
19. Itzykson, C., Zuber, J.B.: The planar approximation II. *J. Math. Phys.* **21**, 411–421 (1980)
20. Johansson, K.: Non-intersecting paths, random tilings and random matrices. *Probab. Theo Related Fields* **123**, 225–280 (2002)
21. Karlin, S., McGregor, J.: Coincidence probabilities. *Pacific J. Math.* **9**, 1141–1164 (1959)
22. Katori, M., Tanemura, H.: Scaling limit of vicious walks and two-matrix model. *Phys. Rev. E* **66**, Art. No. 011105 (2002)
23. Kuijlaars, A.B.J.: Riemann-Hilbert analysis for orthogonal polynomials. In: *Orthogonal Polynomials and Special Functions* E. Koelink, W. Van Assche, (eds), *Lecture Notes in Mathematics*, Vol. **1817**, Berlin-Heiderberg-New York, Springer-Verlag, 2003, pp. 167–210
24. Kuijlaars, A.B.J., Van Assche, W., Wielonsky, F.: Quadratic Hermite-Padé approximation to the exponential function: a Riemann-Hilbert approach. <http://Constr.org/list/math.CA/0302357>, 2003 *Approx.* **21**, 351–412 (2005)
25. Kuijlaars, A.B.J., Vanlessen, M.: Universality for eigenvalue correlations from the modified Jacobi unitary ensemble. *Internat. Math. Research Notices* 2002, 1575–1600 (2002)
26. Kuijlaars, A.B.J., Vanlessen, M.: Universality for eigenvalue correlations at the origin of the spectrum. *Commun. Math. Phys.* **243**, 163–191 (2003)
27. Mehta, M.L.: *Random Matrices*. 2nd edition, Boston: Academic Press, 1991
28. Nagao, T., Forrester, P.J.: Vicious random walkers and a discretization of Gaussian random matrix ensembles. *Nuclear Phys. B* **620**, 551–565 (2002)
29. Pastur, L.A.: The spectrum of random matrices (Russian). *Teoret. Mat. Fiz.* **10**, 102–112 (1972)
30. Tracy, C.A., Widom, H.: The Pearcey process. <http://arxiv.org/abs/math.PR/0412005>, 2004
31. Van Assche, W., Coussement, E.: Some classical multiple orthogonal polynomials. *J. Comput. Appl. Math.* **127**, 317–347 (2001)
32. Van Assche, W., Geronimo, J.S., Kuijlaars, A.B.J.: Riemann-Hilbert problems for multiple orthogonal polynomials. In: *Special Functions 2000: Current Perspectives and Future Directions*, J. Bustoz et al., (eds), Dordrecht: Kluwer, 2001, pp. 23–59
33. Zinn-Justin, P.: Random Hermitian matrices in an external field. *Nucl. Phys. B* **497**, 725–732 (1997)
34. Zinn-Justin, P.: Universality of correlation functions of Hermitian random matrices in an external field. *Commun. Math. Phys.* **194**, 631–650 (1998)