

Large n Limit of Gaussian Random Matrices with External Source, Part I

Pavel Bleher^{1,*}, Arno B. J. Kuijlaars^{2,**}

¹ Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 N. Blackford St., Indianapolis, IN 46202, U.S.A. E-mail: bleher@math.iupui.edu

² Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, 3001 Leuven, Belgium. E-mail: arno@wis.kuleuven.ac.be

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Dedicated to Freeman Dyson on his eightieth birthday

Abstract: We consider the random matrix ensemble with an external source

$$\frac{1}{Z_n} e^{-n\text{Tr}(\frac{1}{2}M^2 - AM)} dM$$

defined on $n \times n$ Hermitian matrices, where A is a diagonal matrix with only two eigenvalues $\pm a$ of equal multiplicity. For the case $a > 1$, we establish the universal behavior of local eigenvalue correlations in the limit $n \rightarrow \infty$, which is known from unitarily invariant random matrix models. Thus, local eigenvalue correlations are expressed in terms of the sine kernel in the bulk and in terms of the Airy kernel at the edge of the spectrum. We use a characterization of the associated multiple Hermite polynomials by a 3×3 -matrix Riemann-Hilbert problem, and the Deift/Zhou steepest descent method to analyze the Riemann-Hilbert problem in the large n limit.

1. Introduction and Statement of Results

We will consider the random matrix ensemble with an external source,

$$\mu_n(dM) = \frac{1}{Z_n} e^{-n\text{Tr}(V(M) - AM)} dM, \quad (1.1)$$

defined on $n \times n$ Hermitian matrices M . The number n is a large parameter in the ensemble. The Gaussian ensemble, $V(M) = \frac{1}{2}M^2$, has been solved in the papers of Pastur [26] and Brézin-Hikami [9–13], by using spectral methods and a contour integration formula for the determinantal kernel. In the present work we will develop a completely different approach to the solution of the Gaussian ensemble with external source. Our approach

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is based on the Riemann-Hilbert problem and it is applicable, in principle, to a general V .

We will assume that the external source A is a fixed diagonal matrix with n_1 eigenvalues a and n_2 eigenvalues $(-a)$,

$$A = \text{diag}(\underbrace{a, \dots, a}_{n_1}, \underbrace{-a, \dots, -a}_{n_2}), \quad n_1 + n_2 = n. \quad (1.2)$$

As shown by P. Zinn-Justin [29], for any $m \geq 1$, the m -point correlation function of eigenvalues of M has the determinantal form,

$$R_m(\lambda_1, \dots, \lambda_m) = \det(K_n(\lambda_j, \lambda_k))_{1 \leq j, k \leq m}. \quad (1.3)$$

In our previous work [7] we show that the kernel $K_n(x, y)$ can be expressed in terms of a solution to the following matrix Riemann-Hilbert (RH) problem: find $Y : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{3 \times 3}$ such that

- Y is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- for $x \in \mathbb{R}$, we have

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.4)$$

where

$$w_1(x) = e^{-n(V(x)-ax)}, \quad w_2(x) = e^{-n(V(x)+ax)}, \quad (1.5)$$

and $Y_+(x)$ ($Y_-(x)$) denotes the limit of $Y(z)$ as $z \rightarrow x$ from the upper (lower) half-plane,

- as $z \rightarrow \infty$, we have

$$Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}, \quad (1.6)$$

where I denotes the 3×3 identity matrix.

Namely,

$$\begin{aligned} K_n(x, y) &= e^{-\frac{1}{2}n(V(x)+V(y))} \frac{e^{nay}[Y(y)^{-1}Y(x)]_{21} + e^{-nay}[Y(y)^{-1}Y(x)]_{31}}{2\pi i(x-y)} \\ &= \frac{e^{-\frac{1}{2}n(V(x)+V(y))}}{2\pi i(x-y)} (0 \ e^{nay} \ e^{-nay}) Y(y)^{-1} Y(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (1.7)$$

The RH problem has a unique solution and the solution is expressed in terms of multiple orthogonal polynomials, see [7] and Sect. 2.1 below. For now, let us mention that the $(1, 1)$ entry Y_{11} satisfies

$$Y_{11}(z) = \mathbb{E}[\det(zI - M)], \quad (1.8)$$

where \mathbb{E} denotes expectation with respect to the measure (1.1). So it is the average characteristic polynomial for the random matrix ensemble.

It is the aim of this paper to analyze the RH problem as $n \rightarrow \infty$, by using the method of steepest descent / stationary phase of Deift and Zhou [17]. We focus here on the Gaussian case $V(x) = \frac{1}{2}x^2$. Our first result concerns the limiting mean eigenvalue density.

Theorem 1.1. *Let $V(M) = \frac{1}{2}M^2$, $n_1 = n_2 = n/2$ (so n is even) and let $a > 1$. Then the limiting mean density of eigenvalues*

$$\rho(x) = \lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) \tag{1.9}$$

exists, and it is supported by two intervals, $[-z_1, -z_2]$ and $[z_2, z_1]$. The density $\rho(x)$ is expressed as

$$\rho(x) = \frac{1}{\pi} |\operatorname{Im} \xi(x)|, \tag{1.10}$$

where $\xi = \xi(x)$ solve the cubic equation,

$$\xi^3 - x\xi^2 - (a^2 - 1)\xi + xa^2 = 0 \tag{1.11}$$

(Pastur's equation). The density ρ is real analytic on $(-z_1, -z_2) \cup (z_2, z_1)$ and it vanishes like a square root at the edge points of its support, i.e., there exist constants $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned} \rho(x) &= \frac{\rho_j}{\pi} |x - z_j|^{1/2} (1 + o(1)) \quad \text{as } x \rightarrow z_j, x \in (z_2, z_1), \\ \rho(x) &= \frac{\rho_j}{\pi} |x + z_j|^{1/2} (1 + o(1)) \quad \text{as } x \rightarrow -z_j, x \in (-z_1, -z_2). \end{aligned} \tag{1.12}$$

Remark. We obtain ρ from an analysis of the equation

$$z = \frac{\xi^3 - (a^2 - 1)\xi}{\xi^2 - a^2}. \tag{1.13}$$

The critical points of the mapping (1.13) satisfy

$$\xi^2 = \frac{1}{2} + a^2 \pm \frac{1}{2} \sqrt{1 + 8a^2}. \tag{1.14}$$

For $a > 1$, the four critical points are real, and they correspond to four real branch points $\pm z_1, \pm z_2$ with $z_1 > z_2 > 0$. We denote the three inverses of (1.13) by $\xi_j(z)$, $j = 1, 2, 3$, where ξ_1 is chosen such that $\xi_1(z) \sim z$ as $z \rightarrow \infty$. Then ξ_1 has an analytic continuation to $\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])$ and $\operatorname{Im} \xi_{1+}(x) > 0$ for $x \in (-z_1, -z_2) \cup (z_2, z_1)$. Then the density ρ is

$$\rho(x) = \frac{1}{\pi} \operatorname{Im} \xi_{1+}(x), \tag{1.15}$$

see Sect. 3.

The assumption $a > 1$ is essential for four real branch points and a limiting mean eigenvalue density which is supported on two disjoint intervals. For $0 < a < 1$, two branch points are purely imaginary, and the limiting mean eigenvalue density is supported on one interval. The main theorem on the local eigenvalue correlations continues to hold, but its proof requires a different analysis of the RH problem. This will be done in Part II, see [2]. In Part III we will discuss the case $a = 1$.

Remark. The density ρ can also be characterized by a minimization problem for logarithmic potentials. Consider the following energy functional defined on pairs (μ_1, μ_2) of measures:

$$\begin{aligned} E(\mu_1, \mu_2) &= \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(x) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) \\ &+ \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) + \int \left(\frac{1}{2}x^2 - ax \right) d\mu_1(x) \\ &+ \int \left(\frac{1}{2}x^2 + ax \right) d\mu_2(x). \end{aligned}$$

The problem is to minimize $E(\mu_1, \mu_2)$ among all pairs (μ_1, μ_2) of measures on \mathbb{R} with $\int d\mu_1 = \int d\mu_2 = \frac{1}{2}$. There is a unique minimizer, and for $a > 1$, it can be shown that μ_1 is supported on $[z_2, z_1]$, μ_2 is supported on $[-z_1, -z_2]$ and ρ is the density of $\mu_1 + \mu_2$. This minimal energy problem is similar to the minimal energy problem for Angelesco systems in the theory of multiple orthogonal polynomials, see [4, 19].

It is possible to base the asymptotic analysis of the RH problem on the minimization problem, as done by Deift et al, see [14–16], for the unitarily invariant random matrix model. However, we will not pursue that here.

Our main results concern the universality of local eigenvalue correlations in the large n limit. This was established for unitarily invariant random matrix models

$$\frac{1}{Z_n} e^{-n\text{Tr}V(M)} dM \tag{1.16}$$

by Bleher and Its [5] for a quartic polynomial V , and by Deift et al [15] for general real analytic V . The universality may be expressed by the following limit

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho(x_0)} K_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \frac{\sin \pi(u-v)}{\pi(u-v)} \tag{1.17}$$

which is valid for x_0 in the bulk of the spectrum, i.e., for x_0 such that the limiting mean eigenvalue density $\rho(x_0)$ is positive. The proof of (1.17) established Dyson's universality conjecture [18, 23] for unitary ensembles.

In our case, we use a rescaled version of the kernel K_n ,

$$\hat{K}_n(x, y) = e^{n(h(x)-h(y))} K_n(x, y) \tag{1.18}$$

for some function h . The rescaling (1.18) is allowed because it does not affect the correlation functions R_m (1.3), which are expressed as determinants of the kernel. Note that the kernel K_n of (1.7) is non-symmetric and there is no obvious a priori scaling for it. The function h in (1.18) has the following form on $(-z_1, -z_2) \cup (z_2, z_1)$:

$$h(x) = -\frac{1}{4}x^2 + \text{Re } \lambda_{1+}(x), \quad x \in (-z_1, -z_2) \cup (z_2, z_1) \tag{1.19}$$

with $\lambda_{1+}(x) = \int_{z_1}^x \xi_{1+}(s) ds$, where ξ_1 is as in the first remark after Theorem 1.1.

Theorem 1.2. *Let $V(M) = \frac{1}{2}M^2$, $n_1 = n_2 = n/2$, and let $a > 1$. Let z_1, z_2 and ρ be as in Theorem 1.1 and let \hat{K}_n be as in (1.18). Then for every $x_0 \in (-z_1, -z_2) \cup (z_2, z_1)$ and $u, v \in \mathbb{R}$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n\rho(x_0)} \hat{K}_n \left(x_0 + \frac{u}{n\rho(x_0)}, x_0 + \frac{v}{n\rho(x_0)} \right) = \frac{\sin \pi(u - v)}{\pi(u - v)}. \tag{1.20}$$

Our final result concerns the eigenvalue correlations near the edge points $\pm z_j$. For unitarily invariant random matrix ensembles (1.16) the local correlations near edge points are expressed in the limit $n \rightarrow \infty$ in terms of the Airy kernel

$$\frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u - v}, \tag{1.21}$$

provided that the limiting mean eigenvalue density vanishes like a square root, which is generically the case [21]. In our non-unitarily invariant random matrix model, the limiting mean eigenvalue density vanishes like a square root, (1.12), and indeed we recover the kernel (1.21) in the limit $n \rightarrow \infty$.

Theorem 1.3. *We use the same notation as in Theorem 1.2. Let ρ_1 and ρ_2 be the constants from (1.12). Then for every $u, v \in \mathbb{R}$ we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{(\rho_1 n)^{2/3}} \hat{K}_n \left(z_1 + \frac{u}{(\rho_1 n)^{2/3}}, z_1 + \frac{v}{(\rho_1 n)^{2/3}} \right) \\ = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u - v}, \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{(\rho_2 n)^{2/3}} \hat{K}_n \left(z_2 - \frac{u}{(\rho_2 n)^{2/3}}, z_2 - \frac{v}{(\rho_2 n)^{2/3}} \right) \\ = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u - v}. \end{aligned} \tag{1.23}$$

Similar limits hold near the edge points $-z_1$ and $-z_2$.

As said before, our results follow from an asymptotic analysis of the RH problem (1.4)–(1.6), which involves 3×3 matrices. In the past, asymptotics for RH problems has mostly been restricted to 2×2 -matrix valued RH problems, see e.g. [5, 6, 15, 16] and references cited therein. The first asymptotic analysis of a 3×3 matrix RH problem appeared in [22] in an approximation problem for the exponential function. In the present work we use some of the ideas of [22].

As in [22] a main tool in the analysis is an appropriate three sheeted Riemann surface. To motivate the choice of the Riemann surface we describe in Sect. 2 the recurrence relations and differential equations that are satisfied by a matrix Ψ , which is an easy modification of Y , see (2.7) below. The Riemann surface is studied in Sect. 3 and we obtain from it the functions ξ_j and λ_j , $j = 1, 2, 3$, that are necessary for the transformations of the RH problem. The first transformation $Y \mapsto T$ normalizes the RH problem at infinity and at the same time introduces oscillating diagonal entries in the jump matrices on the cuts $[-z_1, -z_2]$ and $[z_2, z_1]$, see Sect. 4. The second transformation $T \mapsto S$ involves opening of lenses around the cuts, which results in a RH problem for S with rapidly decaying off-diagonal entries in the jump matrices on the upper and lower boundaries

of the lenses, see Sect. 5. The next step is the construction of a parametrix, an approximate solution to the RH problem. In Sect. 6 we ignore all jumps in the RH problem for S , except those on the cuts $[-z_1, -z_2], [z_2, z_1]$. This leads to a model RH problem, which we solve by lifting it to the Riemann surface via the functions ξ_k . This leads to the parametrix away from the edge points $\pm z_1, \pm z_2$. A separate construction is needed near the edge points. This is done in Sect. 7 where we build the local parametrices out of Airy functions. The final transformation $S \mapsto R$ is done in Sect. 8 and it leads to a RH problem for R whose jump matrices are uniformly close to the identity matrix. Then we can use estimates on solutions of RH problems, see [14], to conclude that R is close to the identity matrix, with error estimates. Having that we can give the proofs of the theorems in Sect. 9.

Our approach proves simultaneously large n asymptotics of the $(1, 1)$ entry of Y , which by (1.8) is equal to the average characteristic polynomial. This polynomial is called a multiple Hermite polynomial for the case of $V(x) = \frac{1}{2}x^2$, see [7] and Sect. 2 below. Since its asymptotics may be of independent interest, we consider it briefly in Sect. 10 below. More information on multiple orthogonal polynomials and their asymptotics can be found in [19, 24, 25], see also the surveys [1, 4] and the references cited therein.

2. Recurrence Relations and Differential Equations

In order to motivate the introduction of the Riemann surface associated with (1.13) we discuss here the recurrence relations and differential equations that are satisfied by the solution of the Riemann-Hilbert problem (1.4)–(1.6) in case $V(x) = \frac{1}{2}x^2$. It also reveals the integrable structure. We note however, that the results of this section are not essential for the rest of the paper.

For the recurrence relations we need to separate the indices n_1 and n_2 in the asymptotic behavior (1.6) from the exponent n in the weight functions w_1, w_2 of (1.5). In this section we put

$$w_1(x) = e^{-N(\frac{1}{2}x^2 - ax)}, \quad w_2(x) = e^{-N(\frac{1}{2}x^2 + ax)}, \quad (2.1)$$

where N is fixed, and we let $Y = Y_{n_1, n_2}$ be the solution of the Riemann-Hilbert problem (1.4), (1.6) with $V(x) = \frac{1}{2}x^2$ and w_1, w_2 given by (2.1). Let $P_{n_1, n_2}(x) = x^n + \dots$ be a monic polynomial of degree $n = n_1 + n_2$ such that for $k = 1, 2$,

$$\int_{-\infty}^{\infty} P_{n_1, n_2}(x) x^j w_k(x) dx = 0, \quad 0 \leq j \leq n_k - 1. \quad (2.2)$$

The polynomial $P_{n_1, n_2}(x)$ is unique and it is called a multiple Hermite polynomial, see [3, 8, 27]. Denote for $k = 1, 2$,

$$h_{n_1, n_2}^{(k)} = \int_{-\infty}^{\infty} P_{n_1, n_2}(x) x^{n_k} w_k(x) dx \neq 0. \quad (2.3)$$

The solution to the RH problem is

$$Y_{n_1, n_2} = \begin{pmatrix} P_{n_1, n_2} & C(P_{n_1, n_2} w_1) & C(P_{n_1, n_2} w_2) \\ c_1 P_{n_1-1, n_2} & c_1 C(P_{n_1-1, n_2} w_1) & c_1 C(P_{n_1-1, n_2} w_2) \\ c_2 P_{n_1, n_2-1} & c_2 C(P_{n_1, n_2-1} w_1) & c_2 C(P_{n_1, n_2-1} w_2) \end{pmatrix}, \quad (2.4)$$

with the constants

$$c_1 = -\frac{2\pi i}{h_{n_1-1, n_2}^{(1)}}, \quad c_2 = -\frac{2\pi i}{h_{n_1, n_2-1}^{(2)}}, \quad (2.5)$$

and where Cf denotes the Cauchy transform of f ,

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(s)}{s-z} ds. \quad (2.6)$$

The recurrence relations and differential equations are more nicely formulated in terms of the function

$$\begin{aligned} \Psi_{n_1, n_2} &= \begin{pmatrix} P_{n_1, n_2} e^{-\frac{1}{2}Nz^2} & C(P_{n_1, n_2} w_1) e^{-Naz} & C(P_{n_1, n_2} w_2) e^{Naz} \\ P_{n_1-1, n_2} e^{-\frac{1}{2}Nz^2} & C(P_{n_1-1, n_2} w_1) e^{-Naz} & C(P_{n_1-1, n_2} w_2) e^{Naz} \\ P_{n_1, n_2-1} e^{-\frac{1}{2}Nz^2} & C(P_{n_1, n_2-1} w_1) e^{-Naz} & C(P_{n_1, n_2-1} w_2) e^{Naz} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1^{-1} & 0 \\ 0 & 0 & c_2^{-1} \end{pmatrix} Y_{n_1, n_2} \begin{pmatrix} e^{-\frac{1}{2}Nz^2} & 0 & 0 \\ 0 & e^{-Naz} & 0 \\ 0 & 0 & e^{Naz} \end{pmatrix}. \end{aligned} \quad (2.7)$$

The function $\Psi = \Psi_{n_1, n_2}$ solves the following RH problem:

- Ψ is analytic on $\mathbb{C} \setminus \mathbb{R}$,
- for $x \in \mathbb{R}$, we have

$$\Psi_+(x) = \Psi_-(x) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.8)$$

- as $z \rightarrow \infty$, we have

$$\Psi(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n e^{-\frac{1}{2}Nz^2} & 0 & 0 \\ 0 & c_1^{-1} z^{-n_1} e^{-Naz} & 0 \\ 0 & 0 & c_2^{-1} z^{-n_2} e^{Naz} \end{pmatrix}. \quad (2.9)$$

Proposition 2.1. *We have the recurrence relations,*

$$\begin{aligned} \Psi_{n_1+1, n_2}(z) &= \begin{pmatrix} z-a-\frac{n_1}{N} & -\frac{n_2}{N} \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \Psi_{n_1, n_2}(z), \\ \Psi_{n_1, n_2+1}(z) &= \begin{pmatrix} z+a-\frac{n_1}{N} & -\frac{n_2}{N} \\ 1 & 2a \\ 1 & 0 \end{pmatrix} \Psi_{n_1, n_2}(z), \end{aligned} \quad (2.10)$$

and the differential equation,

$$\Psi'_{n_1, n_2}(z) = N \begin{pmatrix} -z & \frac{n_1}{N} & \frac{n_2}{N} \\ -1 & -a & 0 \\ -1 & 0 & a \end{pmatrix} \Psi_{n_1, n_2}(z). \quad (2.11)$$

The proof of Proposition 2.1 is given in Appendix C below.

We look for a WKB solution of the differential equation (2.11) of the form

$$\Psi_{n_1, n_2}(z) = W(z)e^{-N\Lambda(z)}, \quad (2.12)$$

where Λ is a diagonal matrix. By substituting this form into (2.11) we obtain the equation,

$$-W\Lambda'W^{-1} = A + \frac{1}{N}W'W^{-1}, \quad (2.13)$$

where A is the matrix of coefficients in (2.11). By dropping the last term we reduce it to the eigenvalue problem,

$$W\Lambda'W^{-1} = -A. \quad (2.14)$$

The characteristic polynomial is

$$\begin{aligned} \det[\xi I + A] &= \begin{vmatrix} \xi - z & t_1 & t_2 \\ -1 & \xi - a & 0 \\ -1 & 0 & \xi + a \end{vmatrix} \\ &= \xi^3 - z\xi^2 + (t_1 + t_2 - a^2)\xi + (t_1 - t_2 + za)a, \end{aligned} \quad (2.15)$$

where $t_1 = \frac{n_1}{N}$ and $t_2 = \frac{n_2}{N}$.

The spectral curve $\xi^3 - z\xi^2 + (t_1 + t_2 - a^2)\xi + (t_1 - t_2 + za)a = 0$ defines a Riemann surface, which in the case of interest in this paper (where $N = n$ and $n_1 = n_2 = \frac{1}{2}n$) reduces to

$$\xi^3 - z\xi^2 - (a^2 - 1)\xi + za^2 = 0. \quad (2.16)$$

This defines the Riemann surface that will play a central role in the rest of the paper.

3. Riemann Surface

The Riemann surface is given by Eq. (2.16) or, if we solve for z ,

$$z = \frac{\xi^3 - (a^2 - 1)\xi}{\xi^2 - a^2}. \quad (3.1)$$

There are three inverse functions to (3.1), which we choose such that as $z \rightarrow \infty$,

$$\begin{aligned} \xi_1(z) &= z - \frac{1}{z} + O\left(\frac{1}{z^3}\right), \\ \xi_2(z) &= a + \frac{1}{2z} + O\left(\frac{1}{z^2}\right), \\ \xi_3(z) &= -a + \frac{1}{2z} + O\left(\frac{1}{z^2}\right). \end{aligned} \quad (3.2)$$

We need to find the sheet structure of the Riemann surface (2.16). The critical points of $z(\xi)$ satisfy the equation

$$\xi^4 - (1 + 2a^2)\xi^2 + (a^2 - 1)a^2 = 0, \quad (3.3)$$

which is biquadratic. The roots are

$$\xi_{1,2}^2 = \frac{1}{2} + a^2 \pm \frac{1}{2}\sqrt{1 + 8a^2}. \quad (3.4)$$

The value $a = 1$ is critical, in the sense that for $a > 1$ all the roots are real, while for $a < 1$, two are real and two are purely imaginary. In this paper we will consider the case $a > 1$. As noted before, we will consider the cases $a < 1$ and $a = 1$ in Parts II and III.

Set

$$p, q = \sqrt{\frac{1}{2} + a^2 \mp \frac{1}{2}\sqrt{1 + 8a^2}}, \quad 0 < p < q. \quad (3.5)$$

Then the critical points are $\xi = \pm p, \pm q$. The branch points on the z -plane are $\pm z_1$ and $\pm z_2$, where

$$z_1 = q \frac{\sqrt{1 + 8a^2} + 3}{\sqrt{1 + 8a^2} + 1}, \quad z_2 = p \frac{\sqrt{1 + 8a^2} - 3}{\sqrt{1 + 8a^2} - 1}, \quad 0 < z_2 < z_1. \quad (3.6)$$

We can show that ξ_1, ξ_2 , and ξ_3 have analytic extensions to $\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])$, $\mathbb{C} \setminus [z_2, z_1]$ and $\mathbb{C} \setminus [-z_1, -z_2]$, respectively. Also on the cut $[z_2, z_1]$,

$$\begin{aligned} \xi_{1+}(x) &= \overline{\xi_{1-}(x)} = \xi_{2-}(x) = \overline{\xi_{2+}(x)}, \quad z_2 \leq x \leq z_1, \\ \text{Im } \xi_{1+}(x) &> 0, \quad z_2 < x < z_1, \end{aligned} \quad (3.7)$$

and $\xi_3(x)$ is real. On the cut $[-z_1, -z_2]$,

$$\begin{aligned} \xi_{1+}(x) &= \overline{\xi_{1-}(x)} = \xi_{3-}(x) = \overline{\xi_{3+}(x)}, \quad -z_1 \leq x \leq -z_2, \\ \text{Im } \xi_{1+}(x) &> 0, \quad -z_1 < x < -z_2, \end{aligned} \quad (3.8)$$

and $\xi_2(x)$ is real. Figure 1 depicts the three sheets of the Riemann surface (2.16).

We define

$$\rho(x) = \frac{1}{\pi} \text{Im } \xi_{1+}(x), \quad x \in [-z_1, -z_2] \cup [z_2, z_1]. \quad (3.9)$$

Proposition 3.1. *We have $\rho(x) > 0$ for $x \in (-z_1, -z_2) \cup (z_2, z_1)$ and*

$$\int_{-z_1}^{-z_2} \rho(x) dx = \int_{z_2}^{z_1} \rho(x) dx = \frac{1}{2}. \quad (3.10)$$

Moreover, there are $\rho_1, \rho_2 > 0$ such that

$$\begin{aligned} \rho(x) &= \frac{\rho_j}{\pi} |x - z_j|^{1/2} (1 + O(x - z_j)) \quad \text{as } x \rightarrow z_j, \quad x \in (z_2, z_1), \\ \rho(x) &= \frac{\rho_j}{\pi} |x + z_j|^{1/2} (1 + O(x + z_j)) \quad \text{as } x \rightarrow -z_j, \quad x \in (-z_1, -z_2). \end{aligned} \quad (3.11)$$

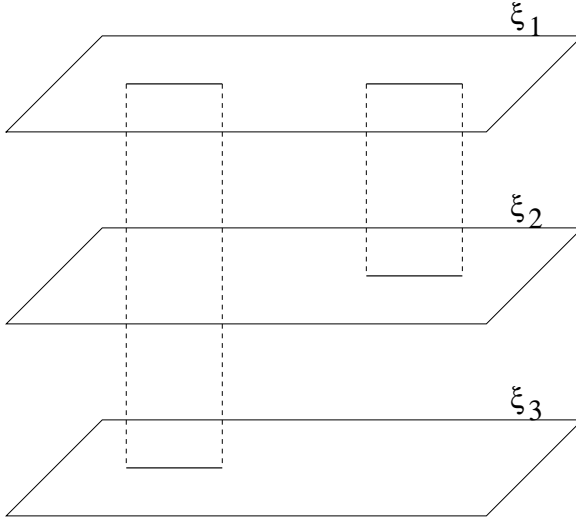


Fig. 1. The three sheets of the Riemann surface

Proof. The fact that $\rho(x) > 0$ for $x \in (-z_1, -z_2) \cup (z_2, z_1)$ was already noted in (3.7) and (3.8).

We have for $x \in [z_2, z_1]$,

$$\rho(x) = \frac{1}{2\pi} \operatorname{Im} (\xi_{1+}(x) - \xi_{1-}(x)) = \frac{1}{2\pi i} (\xi_{1+}(x) - \xi_{1-}(x)) = \frac{1}{2\pi i} (\xi_{2-}(x) - \xi_{2+}(x)).$$

Thus

$$\int_{z_2}^{z_1} \rho(x) dx = \frac{1}{2\pi i} \oint_{\gamma} \xi_2(z) dz,$$

where γ is a contour encircling the interval $[z_2, z_1]$ once in the positive direction. Letting the contour go to infinity and using the asymptotic behavior (3.2) of $\xi_2(z)$ as $z \rightarrow \infty$, we find the value of the second integral in (3.10). The value of the first integral follows in the same way.

For (3.11) we note that near the branch point z_1 , we have for a constant $\rho_1 > 0$,

$$\begin{aligned} \xi_1(z) &= q + \rho_1(z - z_1)^{1/2} + O(z - z_1), \\ \xi_2(z) &= q - \rho_1(z - z_1)^{1/2} + O(z - z_1) \end{aligned} \quad (3.12)$$

as $z \rightarrow z_1$. Similarly, near z_2 we have for a constant $\rho_2 > 0$,

$$\begin{aligned} \xi_1(z) &= p - \rho_2(z_2 - z)^{1/2} + O(z_2 - z), \\ \xi_2(z) &= p + \rho_2(z_2 - z)^{1/2} + O(z_2 - z) \end{aligned} \quad (3.13)$$

as $z \rightarrow z_2$ (with main branches of the square root). By symmetry, we have similar expressions near $-z_1$ and $-z_2$ and (3.11) follows. \square

Next, we need the integrals of the ξ -functions,

$$\lambda_k(z) = \int^z \xi_k(s) ds, \quad k = 1, 2, 3, \tag{3.14}$$

which we take so that λ_1 and λ_2 are analytic on $\mathbb{C} \setminus (-\infty, z_1]$ and λ_3 is analytic on $\mathbb{C} \setminus (-\infty, -z_2]$. From (3.2) it follows that, as $z \rightarrow \infty$,

$$\begin{aligned} \lambda_1(z) &= \frac{z^2}{2} - \ln z + l_1 + O\left(\frac{1}{z^2}\right), \\ \lambda_2(z) &= az + \frac{1}{2} \ln z + l_2 + O\left(\frac{1}{z}\right), \\ \lambda_3(z) &= -az + \frac{1}{2} \ln z + l_3 + O\left(\frac{1}{z}\right), \end{aligned} \tag{3.15}$$

where l_1, l_2, l_3 are some constants, which we choose as follows. We choose l_1 and l_2 such that

$$\lambda_1(z_1) = \lambda_2(z_1) = 0,$$

and then l_3 such that

$$\lambda_3(-z_2) = \lambda_{1+}(-z_2) = \lambda_{1-}(-z_2) - \pi i.$$

Then we have the following jump relations:

$$\begin{aligned} \lambda_{1+}(x) - \lambda_{1-}(x) &= \pi i, & x \in [-z_2, z_2], \\ \lambda_{1+}(x) - \lambda_{1-}(x) &= -2\pi i, & x \in (-\infty, -z_1], \\ \lambda_{2+}(x) - \lambda_{2-}(x) &= \pi i, & x \in (-\infty, z_2], \\ \lambda_{1+}(x) &= \lambda_{2-}(x), \quad \lambda_{1-}(x) = \lambda_{2+}(x), & x \in [z_2, z_1], \\ \lambda_{1+}(x) &= \lambda_{3-}(x), \quad \lambda_{1-}(x) - \pi i = \lambda_{3+}(x), & x \in [-z_1, -z_2]. \\ \lambda_{3+}(x) - \lambda_{3-}(x) &= \pi i, & x \in (-\infty, -z_1]. \end{aligned} \tag{3.16}$$

Note that due to the first two equations of (3.16) we have that $e^{n\lambda_1(z)}$ is analytic on the complex plane with cuts on $[-z_1, -z_2]$ and $[z_2, z_1]$ (recall that n is even). Furthermore, we also see that $e^{n\lambda_2(z)}$ (resp., $e^{n\lambda_3(z)}$) is analytic on the complex plane with a cut on $[z_2, z_1]$ (resp., $[-z_1, -z_2]$), see Fig. 1.

For later use, we state the following two propositions.

Proposition 3.2. *On $\mathbb{R} \setminus [z_2, z_1]$ we have $\operatorname{Re} \lambda_{2+} < \operatorname{Re} \lambda_{1-}$, and on $\mathbb{R} \setminus [-z_1, -z_2]$, we have $\operatorname{Re} \lambda_{3+} < \operatorname{Re} \lambda_{1-}$.*

Proof. It is easy to see that $\xi_1(x) > \xi_2(x)$ for $x > z_1$. Since $\lambda_1(z_1) = \lambda_2(z_1)$ and $\lambda'_j = \xi_j$ for $j = 1, 2, 3$, it is then clear that $\lambda_1(x) > \lambda_2(x)$ for $x > z_1$.

We also have that $\operatorname{Re} \xi_{1-}(x) < \xi_2(x)$ for $x < z_2$, from which it follows that $\operatorname{Re} \lambda_{1-}(x) > \operatorname{Re} \lambda_{2+}(x)$.

Similarly we find that $\operatorname{Re} \lambda_{3+} < \operatorname{Re} \lambda_{1-}$ on $\mathbb{R} \setminus [-z_1, -z_2]$. \square

Proposition 3.3. (a) *The open interval (z_2, z_1) has a neighborhood U_1 in the complex plane such that*

$$\operatorname{Re} \lambda_3(z) < \operatorname{Re} \lambda_1(z) < \operatorname{Re} \lambda_2(z)$$

for every $z \in U_1 \setminus (z_2, z_1)$.

(b) *The open interval $(-z_1, -z_2)$ has a neighborhood U_2 in the complex plane such that*

$$\operatorname{Re} \lambda_2(z) < \operatorname{Re} \lambda_1(z) < \operatorname{Re} \lambda_3(z)$$

for every $z \in U_2 \setminus (-z_1, -z_2)$.

Proof. The function $F = \lambda_{2+} - \lambda_{1+}$ is purely imaginary on (z_2, z_1) . Its derivative is $F'(x) = \xi_{2+}(x) - \xi_{1+}(x) = -2\pi i \rho(x)$, and this has negative imaginary part. The Cauchy Riemann equations then imply that the real part of F increases as we move from the interval (z_2, z_1) into the upper half-plane. Thus $\operatorname{Re} \lambda_2(z) - \operatorname{Re} \lambda_1(z) > 0$ for z near (z_2, z_1) in the upper half-plane. Similarly, $\operatorname{Re} \lambda_2(z) - \operatorname{Re} \lambda_1(z) > 0$ for z near (z_2, z_1) in the lower half-plane.

By Proposition 3.2 we have $\operatorname{Re} \lambda_3 < \operatorname{Re} \lambda_{1-}$ on $[z_2, z_1]$. By continuity, the inequality continues to hold in a complex neighborhood of $[z_2, z_1]$. This proves part (a). The proof of part (b) is similar. \square

4. First Transformation of the RH Problem

Using the functions λ_j and the constants l_j , $j = 1, 2, 3$, from the previous section, we define

$$T(z) = \operatorname{diag}\left(e^{-nl_1}, e^{-nl_2}, e^{-nl_3}\right) Y(z) \operatorname{diag}\left(e^{n(\lambda_1(z) - \frac{1}{2}z^2)}, e^{n(\lambda_2(z) - az)}, e^{n(\lambda_3(z) + az)}\right). \quad (4.1)$$

Then by (1.4) and (4.1), we have $T_+(x) = T_-(x)j_T(x)$, $x \in \mathbb{R}$, where

$$j_T(x) = \begin{pmatrix} e^{n(\lambda_{1+}(x) - \lambda_{1-}(x))} & e^{n(\lambda_{2+}(x) - \lambda_{1-}(x))} & e^{n(\lambda_{3+}(x) - \lambda_{1-}(x))} \\ 0 & e^{n(\lambda_{2+}(x) - \lambda_{2-}(x))} & 0 \\ 0 & 0 & e^{n(\lambda_{3+}(x) - \lambda_{3-}(x))} \end{pmatrix}. \quad (4.2)$$

The jump relations (3.16) allow us to simplify the jump matrix j_T on the different parts of the real axis. On $[z_2, z_1]$, (4.2) reduces to

$$j_T = \begin{pmatrix} e^{n(\lambda_1 - \lambda_2)_+} & 1 & e^{n(\lambda_3 - \lambda_1 -)} \\ 0 & e^{n(\lambda_1 - \lambda_2)_-} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.3)$$

and on $[-z_1, -z_2]$ to

$$j_T = \begin{pmatrix} e^{n(\lambda_1 - \lambda_3)_+} & e^{n(\lambda_2 + - \lambda_1 -)} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_1 - \lambda_3)_-} \end{pmatrix}. \quad (4.4)$$

On $(-\infty, -z_1] \cup [-z_2, z_2] \cup [z_1, \infty)$, (4.2) reduces to

$$j_T = \begin{pmatrix} 1 & e^{n(\lambda_{2+}-\lambda_{1-})} & e^{n(\lambda_{3+}-\lambda_{1-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in (-\infty, -z_1] \cup [-z_2, z_2] \cup [z_1, \infty). \quad (4.5)$$

The asymptotics of T are, because of (1.6), (3.15), and (4.1),

$$T(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty. \quad (4.6)$$

Thus T solves the following RH problem:

- T is analytic on $\mathbb{C} \setminus \mathbb{R}$,
-

$$T_+(x) = T_-(x)j_T(x), \quad x \in \mathbb{R}, \quad (4.7)$$

- as $z \rightarrow \infty$,

$$T(z) = I + O\left(\frac{1}{z}\right). \quad (4.8)$$

Using (4.1) in (1.7) we see that the kernel K_n can be expressed in terms of T as follows:

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} (0 \ e^{n\lambda_{2+}(y)} \ e^{n\lambda_{3+}(y)}) T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\lambda_{1+}(x)} \\ 0 \\ 0 \end{pmatrix}. \quad (4.9)$$

5. Second Transformation of the RH Problem

The second transformation of the RH problem is opening of lenses. Consider a lens with vertices z_2, z_1 , see Fig. 2. The lens is contained in the neighborhood U_1 of (z_2, z_1) , see Proposition 3.3. We have the factorization,

$$\begin{aligned} & \begin{pmatrix} e^{n(\lambda_{1-}\lambda_{2+})} & 1 & e^{n(\lambda_{3+}-\lambda_{1-})} \\ 0 & e^{n(\lambda_{1-}\lambda_{2-})} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_{1-}\lambda_{2-})} & 1 & -e^{n(\lambda_{3-}\lambda_{2-})} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_{1-}\lambda_{2+})} & 1 & e^{n(\lambda_{3-}\lambda_{2+})} \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.1) \end{aligned}$$

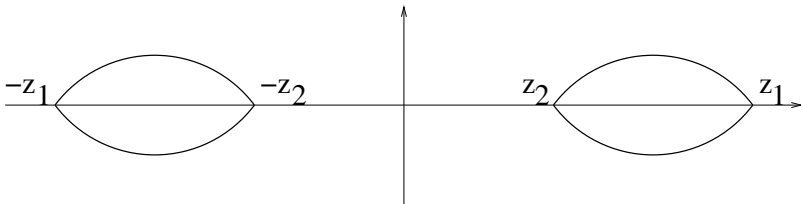


Fig. 2. The lenses with vertices $-z_1, -z_2$ and z_2, z_1

Set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ -e^{n(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{n(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} & \text{in the upper lens region,} \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{n(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} & \text{in the lower lens region.} \end{cases} \quad (5.2)$$

Then (4.7) and (5.2) imply that

$$S_+(x) = S_-(x)j_S(x); \quad j_S(x) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in [z_2, z_1]. \quad (5.3)$$

Similarly, consider a lens with vertices $-z_1, -z_2$, that is contained in U_2 (see Proposition 3.3) and set

$$S(z) = \begin{cases} T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{n(\lambda_1(z)-\lambda_3(z))} & -e^{n(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} & \text{in the upper lens region,} \\ T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1(z)-\lambda_3(z))} & -e^{n(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} & \text{in the lower lens region.} \end{cases} \quad (5.4)$$

Then (4.7) and (5.4) imply that

$$S_+(x) = S_-(x)j_S(x); \quad j_S(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad x \in [-z_1, -z_2]. \quad (5.5)$$

Set

$$S(z) = T(z) \quad \text{outside of the lens regions.} \quad (5.6)$$

Then we have jumps on the boundary of the lenses,

$$S_+(z) = S_-(z)j_S(z), \quad (5.7)$$

where the contours are oriented from left to right (that is, from $-z_1$ to $-z_2$, or from z_2 to z_1), and where S_+ (S_-) denotes the limiting value of S from the left (right) if we traverse the contour according to its orientation. The jump matrix j_S in (5.7) has the form

$$\begin{aligned}
j_S(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1(z)-\lambda_2(z))} & 1 & e^{n(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the upper boundary of the } [z_2, z_1]\text{-lens,} \\
j_S(z) &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1(z)-\lambda_2(z))} & 1 & -e^{n(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix} \text{ on the lower boundary of the } [z_2, z_1]\text{-lens,} \\
j_S(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1(z)-\lambda_3(z))} & e^{n(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ on the upper boundary of the } [-z_1, -z_2]\text{-lens,} \\
j_S(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{n(\lambda_1(z)-\lambda_3(z))} & -e^{n(\lambda_2(z)-\lambda_3(z))} & 1 \end{pmatrix} \text{ on the lower boundary of the } [-z_1, -z_2]\text{-lens.}
\end{aligned} \tag{5.8}$$

On $(-\infty, z_1] \cup [-z_2, z_2] \cup [z_1, \infty)$, S has the same jump as T , so that

$$S_+(x) = S_-(x)j_S(x); \quad j_S(x) = j_T(x), \quad x \in (-\infty, z_1] \cup [-z_2, z_2] \cup [z_1, \infty). \tag{5.9}$$

Thus, S solves the following RH problem:

- S is analytic on $\mathbb{C} \setminus (\mathbb{R} \cup \Gamma)$, where Γ is the boundary of the lenses,

$$S_+(z) = S_-(z)j_S(z), \quad z \in \mathbb{R} \cup \Gamma, \tag{5.10}$$

- as $z \rightarrow \infty$,

$$S(z) = I + O\left(\frac{1}{z}\right). \tag{5.11}$$

The kernel K_n is expressed in terms of S as follows, see (4.9) and the definitions (5.2) and (5.4). For x and y in (z_2, z_1) we have

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & e^{n\lambda_{2+}(y)} & 0 \\ e^{-n\lambda_{1+}(x)} & e^{-n\lambda_{2+}(x)} & 0 \end{pmatrix} S_+^{-1}(y)S_+(x), \tag{5.12}$$

while for x and y in $(-z_1, -z_2)$ we have

$$K_n(x, y) = \frac{e^{\frac{1}{4}n(x^2-y^2)}}{2\pi i(x-y)} \begin{pmatrix} -e^{n\lambda_{1+}(y)} & 0 & e^{n\lambda_{3+}(y)} \\ 0 & e^{-n\lambda_{1+}(x)} & 0 \end{pmatrix} S_+^{-1}(y)S_+(x). \tag{5.13}$$

Since λ_{1+} and λ_{2+} are complex conjugates on (z_2, z_1) , we can rewrite (5.12) for $x, y \in (z_2, z_1)$ as

$$K_n(x, y) = \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{-ni\text{Im}\lambda_{1+}(y)} & e^{-ni\text{Im}\lambda_{1+}(y)} & 0 \\ e^{-ni\text{Im}\lambda_{1+}(x)} & e^{-ni\text{Im}\lambda_{1+}(x)} & 0 \end{pmatrix} S_+^{-1}(y)S_+(x), \tag{5.14}$$

where $h(x) = -\frac{1}{4}x^2 + \operatorname{Re} \lambda_{1+}(x)$ as in (1.19). Similarly, we have for $x, y \in (-z_1, -z_2)$,

$$K_n(x, y) = \frac{e^{n(h(y)-h(x))}}{2\pi i(x-y)} \begin{pmatrix} -e^{ni\operatorname{Im} \lambda_{1+}(y)} & 0 \\ 0 & e^{-ni\operatorname{Im} \lambda_{1+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-ni\operatorname{Im} \lambda_{1+}(x)} \\ 0 \\ e^{ni\operatorname{Im} \lambda_{1+}(x)} \end{pmatrix}. \quad (5.15)$$

6. Model RH Problem

As $n \rightarrow \infty$, the jump matrix $j_S(z)$ is exponentially close to the identity matrix at every z outside of $[-z_1, -z_2] \cup [z_2, z_1]$. This follows from (5.8) and Proposition 3.3 for z on the boundary of the lenses, and from (5.9), (4.3) and Proposition 3.2 for z on the real intervals $(-\infty, -z_1)$, $(-z_2, z_2)$ and (z_1, ∞) .

In this section we solve the following model RH problem, where we ignore the exponentially small jumps: find $M : \mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1]) \rightarrow \mathbb{C}^{3 \times 3}$ such that

- M is analytic on $\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])$,
-

$$M_+(x) = M_-(x) j_S(x), \quad x \in (-z_1, -z_2) \cup (z_2, z_1), \quad (6.1)$$

- as $z \rightarrow \infty$,

$$M(z) = I + O\left(\frac{1}{z}\right). \quad (6.2)$$

This problem is similar to the RH problem considered in [22, Sect. 6.1]. We also follow a similar method to solve it.

We lift the model RH problem to the Riemann surface of (2.16) with the sheet structure as in Fig. 1. Consider to that end the range of the functions ξ_k on the complex plane,

$$\begin{aligned} \Omega_1 &= \xi_1(\mathbb{C} \setminus ([-z_1, -z_2] \cup [z_2, z_1])), \\ \Omega_2 &= \xi_2(\mathbb{C} \setminus [z_2, z_1]), \\ \Omega_3 &= \xi_3(\mathbb{C} \setminus [-z_1, -z_2]). \end{aligned} \quad (6.3)$$

Then $\Omega_1, \Omega_2, \Omega_3$ give a partition of the complex plane into three regions, see Fig. 3. The regions Ω_2, Ω_3 are bounded, $a \in \Omega_2$, $-a \in \Omega_3$, with the symmetry conditions,

$$\overline{\Omega_2} = \Omega_2, \quad \overline{\Omega_3} = \Omega_3, \quad \Omega_2 = -\Omega_3. \quad (6.4)$$

Denote by Γ_k the boundary of Ω_k , $k = 2, 3$, see Fig. 3. Then we have

$$\begin{aligned} \xi_{1+}([z_2, z_1]) &= \xi_{2-}([z_2, z_1]) = \Gamma_2^+ \equiv \Gamma_2 \cap \{\operatorname{Im} z \geq 0\}, \\ \xi_{1-}([z_2, z_1]) &= \xi_{2+}([z_2, z_1]) = \Gamma_2^- \equiv \Gamma_2 \cap \{\operatorname{Im} z \leq 0\}, \\ \xi_{1+}([-z_1, -z_2]) &= \xi_{3-}([-z_1, -z_2]) = \Gamma_3^+ \equiv \Gamma_3 \cap \{\operatorname{Im} z \geq 0\}, \\ \xi_{1-}([-z_1, -z_2]) &= \xi_{3+}([-z_1, -z_2]) = \Gamma_3^- \equiv \Gamma_3 \cap \{\operatorname{Im} z \leq 0\}. \end{aligned} \quad (6.5)$$

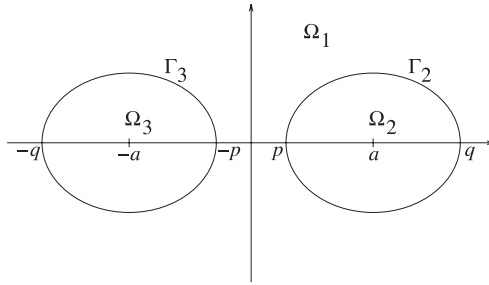


Fig. 3. Partition of the complex ξ -plane

We are looking for a solution M in the following form:

$$M(z) = \begin{pmatrix} M_1(\xi_1(z)) & M_1(\xi_2(z)) & M_1(\xi_3(z)) \\ M_2(\xi_1(z)) & M_2(\xi_2(z)) & M_2(\xi_3(z)) \\ M_3(\xi_1(z)) & M_3(\xi_2(z)) & M_3(\xi_3(z)) \end{pmatrix}, \tag{6.6}$$

where $M_1(\xi), M_2(\xi), M_3(\xi)$ are three analytic functions on $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. To satisfy the jump condition (6.1) we need the following relations for $k = 1, 2, 3$:

$$\begin{aligned} M_{k+}(\xi) &= M_{k-}(\xi), & \xi \in \Gamma_2^- \cup \Gamma_3^-, \\ M_{k+}(\xi) &= -M_{k-}(\xi), & \xi \in \Gamma_2^+ \cup \Gamma_3^+. \end{aligned} \tag{6.7}$$

Since $\xi_1(\infty) = \infty, \xi_2(\infty) = a, \xi_3(\infty) = -a$, then to satisfy (6.2) we demand

$$\begin{aligned} M_1(\infty) &= 1, & M_1(a) &= 0, & M_1(-a) &= 0; \\ M_2(\infty) &= 0, & M_2(a) &= 1, & M_2(-a) &= 0; \\ M_3(\infty) &= 0, & M_3(a) &= 0, & M_3(-a) &= 1. \end{aligned} \tag{6.8}$$

Equations (6.7)–(6.8) have the following solution:

$$M_1(\xi) = \frac{\xi^2 - a^2}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)}}, \quad M_{2,3}(\xi) = c_{2,3} \frac{\xi \pm a}{\sqrt{(\xi^2 - p^2)(\xi^2 - q^2)}}, \tag{6.9}$$

with cuts at Γ_2^+, Γ_3^+ . The constants $c_{2,3}$ are determined by the equations $M_{2,3}(\pm a) = 1$. By (3.3),

$$(\xi^2 - p^2)(\xi^2 - q^2) = \xi^4 - (1 + 2a^2)\xi^2 + (a^2 - 1)a^2, \tag{6.10}$$

hence

$$M_2(a) = c_2 \frac{2a}{\sqrt{-2a^2}}. \tag{6.11}$$

By taking into account the cuts of $M_2(\xi)$ we obtain that

$$M_2(-a) = c_2 i \sqrt{2}, \tag{6.12}$$

hence

$$c_2 = -\frac{i}{\sqrt{2}}. \quad (6.13)$$

Similarly,

$$M_3(a) = c_3 \frac{-2a}{\sqrt{-2a^2}} = c_3 i \sqrt{2}, \quad (6.14)$$

hence c_3 is the same as c_2 ,

$$c_3 = -\frac{i}{\sqrt{2}}. \quad (6.15)$$

Thus, the solution to the model RH problem is given as

$$M(z) = \begin{pmatrix} \frac{\xi_1^2(z)-a^2}{\sqrt{(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & \frac{\xi_2^2(z)-a^2}{\sqrt{(\xi_2^2(z)-p^2)(\xi_2^2(z)-q^2)}} & \frac{\xi_3^2(z)-a^2}{\sqrt{(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \\ -i \frac{\xi_1(z)+a}{\sqrt{2(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & -i \frac{\xi_2(z)+a}{\sqrt{2(\xi_2^2(z)-p^2)(\xi_2^2(z)-q^2)}} & -i \frac{\xi_3(z)+a}{\sqrt{2(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \\ -i \frac{\xi_1(z)-a}{\sqrt{2(\xi_1^2(z)-p^2)(\xi_1^2(z)-q^2)}} & -i \frac{\xi_2(z)-a}{\sqrt{2(\xi_2^2(z)-p^2)(\xi_2^2(z)-q^2)}} & -i \frac{\xi_3(z)-a}{\sqrt{2(\xi_3^2(z)-p^2)(\xi_3^2(z)-q^2)}} \end{pmatrix}, \quad (6.16)$$

with cuts on $[z_2, z_1]$ and $[-z_1, -z_2]$.

The model solution $M(z)$ will be used to construct a *parametrix* for the RH problem for S outside of a small neighborhood of the edge points. Namely, we will fix some $r > 0$ and consider the disks of radius r around the edge points. At the edge points $M(z)$ is not analytic and in a neighborhood of the edge points the parametrix is constructed differently.

7. Parametrix at Edge Points

We consider small disks $D(\pm z_j, r)$ with radius $r > 0$ and centered at the edge points, and look for a local parametrix P defined on the union of the four disks such that

- P is analytic on $D(\pm z_j, r) \setminus (\mathbb{R} \cup \Gamma)$,

$$P_+(z) = P_-(z) j_S(z), \quad z \in (\mathbb{R} \cup \Gamma) \cap D(\pm z_j, r), \quad (7.1)$$

- as $n \rightarrow \infty$,

$$P(z) = \left(I + O\left(\frac{1}{n}\right) \right) M(z) \quad \text{uniformly for } z \in \partial D(\pm z_j, r). \quad (7.2)$$

We consider here the edge point z_1 in detail. We note that by (3.11) and (3.14) we have as $z \rightarrow z_1$,

$$\begin{aligned} \lambda_1(z) &= q(z - z_1) + \frac{2\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^2, \\ \lambda_2(z) &= q(z - z_1) - \frac{2\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^2, \end{aligned} \quad (7.3)$$

so that

$$\lambda_1(z) - \lambda_2(z) = \frac{4\rho_1}{3}(z - z_1)^{3/2} + O(z - z_1)^{5/2}, \quad (7.4)$$

as $z \rightarrow z_1$. Then it follows that

$$\beta(z) = \left[\frac{3}{4}(\lambda_1(z) - \lambda_2(z)) \right]^{2/3} \quad (7.5)$$

is analytic at z_1 , real-valued on the real axis near z_1 and $\beta'(z_1) = \rho_1^{2/3} > 0$. So β is a conformal map from $D(z_1, r)$ to a convex neighborhood of the origin, if r is sufficiently small (which we assume to be the case). We take Γ near z_1 such that

$$\beta(\Gamma \cap D(z_1, r)) \subset \{z \mid \arg(z) = \pm 2\pi/3\}.$$

Then Γ and \mathbb{R} divide the disk $D(z_1, r)$ into four regions numbered I, II, III, and IV, such that $0 < \arg \beta(z) < 2\pi/3$, $2\pi/3 < \arg \beta(z) < \pi$, $-\pi < \arg \beta(z) < -2\pi/3$, and $-2\pi/3 < \arg \beta(z) < 0$ for z in regions I, II, III, and IV, respectively.

Recall that the jumps j_S near z_1 are given by (5.3), (5.8), and (4.3):

$$\begin{aligned} j_S &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1 - r, z_1), \\ j_S &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the upper boundary of the lens in } D(z_1, r), \\ j_S &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & -e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the lower boundary of the lens in } D(z_1, r), \\ j_S &= \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & e^{n(\lambda_3 - \lambda_1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (z_1, z_1 + r]. \end{aligned} \quad (7.6)$$

We write

$$\tilde{P} = \begin{cases} P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_3 - \lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} & \text{in regions I and IV,} \\ P & \text{in regions II and III.} \end{cases} \quad (7.7)$$

Then the jumps for \tilde{P} are $\tilde{P}_+ = \tilde{P}_- j_{\tilde{P}}$, where

$$\begin{aligned}
 j_{\tilde{P}} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [z_1 - r, z_1), \\
 j_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the upper side of the lens in } D(z_1, r), \\
 j_{\tilde{P}} &= \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1 - \lambda_2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on the lower side of the lens in } D(z_1, r), \\
 j_{\tilde{P}} &= \begin{pmatrix} 1 & e^{n(\lambda_2 - \lambda_1)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (z_1, z_1 + r].
 \end{aligned} \tag{7.8}$$

We still have the matching condition

$$\tilde{P}(z) = \left(I + O\left(\frac{1}{n}\right) \right) M(z) \quad \text{uniformly for } z \in \partial D(z_1, r), \tag{7.9}$$

since $\operatorname{Re} \lambda_3 < \operatorname{Re} \lambda_2$ on $\overline{D(z_1, r)}$, which follows from Proposition 3.2.

The RH problem for \tilde{P} is essentially a 2×2 problem, since the jumps (7.8) are non-trivial only in the upper 2×2 block. A solution can be constructed in a standard way out of Airy functions. The Airy function $\operatorname{Ai}(z)$ solves the equation $y'' = zy$ and for any $\varepsilon > 0$, in the sector $\pi + \varepsilon \leq \arg z \leq \pi - \varepsilon$, it has the asymptotics as $z \rightarrow \infty$,

$$\operatorname{Ai}(z) = \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \left(1 + O\left(z^{-3/2}\right) \right). \tag{7.10}$$

The functions $\operatorname{Ai}(\omega z)$, $\operatorname{Ai}(\omega^2 z)$, where $\omega = e^{\frac{2\pi i}{3}}$, also solve the equation $y'' = zy$, and we have the linear relation,

$$\operatorname{Ai}(z) + \omega \operatorname{Ai}(\omega z) + \omega^2 \operatorname{Ai}(\omega^2 z) = 0. \tag{7.11}$$

Write

$$y_0(z) = \operatorname{Ai}(z), \quad y_1(z) = \omega \operatorname{Ai}(\omega z), \quad y_2(z) = \omega^2 \operatorname{Ai}(\omega^2 z), \tag{7.12}$$

and we use these functions to define

$$\Phi(z) = \begin{cases} \begin{pmatrix} y_0(z) - y_2(z) & 0 \\ y'_0(z) - y'_2(z) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } 0 < \arg z < 2\pi/3, \\ \begin{pmatrix} -y_1(z) - y_2(z) & 0 \\ -y'_1(z) - y'_2(z) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } 2\pi/3 < \arg z < \pi, \\ \begin{pmatrix} -y_2(z) & y_1(z) & 0 \\ -y'_2(z) & y'_1(z) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } -\pi < \arg z < -2\pi/3, \\ \begin{pmatrix} y_0(z) & y_1(z) & 0 \\ y'_0(z) & y'_1(z) & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } -2\pi/3 < \arg z < 0. \end{cases} \quad (7.13)$$

Then

$$\tilde{P}(z) = E_n(z)\Phi(n^{2/3}\beta(z))\text{diag}\left(e^{\frac{1}{2}n(\lambda_1(z)-\lambda_2(z))}, e^{-\frac{1}{2}n(\lambda_1(z)-\lambda_2(z))}, 1\right), \quad (7.14)$$

where E_n is an analytic prefactor that takes care of the matching condition (7.9). Explicitly, E_n is given by

$$E_n = \sqrt{\pi}M \begin{pmatrix} 1 & -1 & 0 \\ -i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n^{1/6}\beta^{1/4} & 0 & 0 \\ 0 & n^{-1/6}\beta^{-1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7.15)$$

A similar construction works for a parametrix P around the other edge points.

8. Third Transformation

In the third and final transformation we put

$$\begin{aligned} R(z) &= S(z)M(z)^{-1} \quad \text{for } z \text{ outside the disks } D(\pm z_j, r), \quad j = 1, 2, \\ R(z) &= S(z)P(z)^{-1} \quad \text{for } z \text{ inside the disks.} \end{aligned} \quad (8.1)$$

Then R is analytic on $\mathbb{C} \setminus \Gamma_R$, where Γ_R consists of the four circles $\partial D(\pm z_j, r)$, $j = 1, 2$, the parts of Γ outside the four disks, and the real intervals $(-\infty, -z_1 - r)$, $(-z_2 + r, z_2 - r)$, $(z_1 + r, \infty)$, see Fig. 4. There are jump relations

$$R_+ = R_- j_R, \quad (8.2)$$

where

$$\begin{aligned} j_R &= MP^{-1} \quad \text{on the circles, oriented counterclockwise,} \\ j_R &= Mj_s M^{-1} \quad \text{on the remaining parts of } \Gamma_R. \end{aligned} \quad (8.3)$$

From (7.2) it follows that $j_R = I + O(1/n)$ uniformly on the circles, and from (5.8), (5.9), (4.3) and Propositions 3.2 and 3.3 it follows that $j_R = I + O(e^{-cn})$ for some $c > 0$ as $n \rightarrow \infty$, uniformly on the remaining parts of Γ_R . So we can conclude

$$j_R(z) = I + O\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty, \text{ uniformly on } \Gamma_R. \quad (8.4)$$

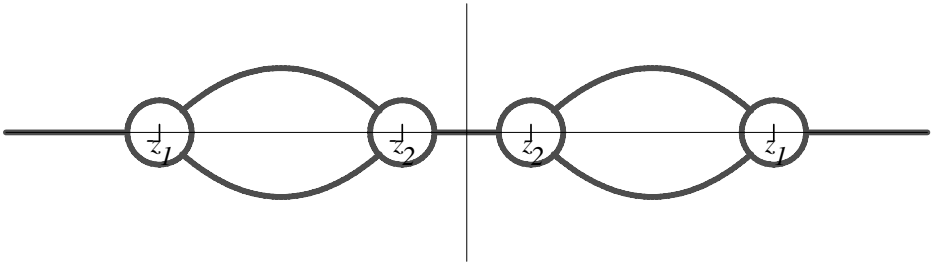


Fig. 4. The contour Γ_R for R

As $z \rightarrow \infty$, we have

$$R(z) = I + O(1/z). \tag{8.5}$$

From (8.2), (8.4), (8.5) and the fact that we can deform the contours in any desired direction, it follows that

$$R(z) = I + O\left(\frac{1}{n(|z| + 1)}\right) \text{ as } n \rightarrow \infty, \tag{8.6}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$, see [14–16, 20].

By Cauchy’s theorem, we then also have

$$R'(z) = O\left(\frac{1}{n(|z| + 1)}\right),$$

and thus

$$R^{-1}(y)R(x) = I + R^{-1}(y) (R(x) - R(y)) = I + O\left(\frac{x - y}{n}\right), \tag{8.7}$$

which is the form we will use below.

9. Proofs of the Theorems

9.1. *Proof of Theorem 1.1.* Consider $x \in (z_2, z_1)$. We may assume that the circles around the edge points are such that x is outside of the four disks. Then (8.1) shows that $S(x) = R(x)M(x)$ and it follows easily from (8.7) and the fact that M_+ is real analytic in a neighborhood of x that

$$S_+^{-1}(y)S_+(x) = I + O(x - y) \text{ as } y \rightarrow x \tag{9.1}$$

uniformly in n . Thus by (5.14) we have that

$$\begin{aligned} K_n(x, y) &= \frac{e^{n(h(y)-h(x))}}{2\pi i(x - y)} \begin{pmatrix} -e^{ni\text{Im } \lambda_{1+}(y)} & e^{-ni\text{Im } \lambda_{1+}(y)} & 0 \end{pmatrix} (I + O(x - y)) \begin{pmatrix} e^{-ni\text{Im } \lambda_{1+}(x)} \\ e^{ni\text{Im } \lambda_{1+}(x)} \\ 0 \end{pmatrix} \\ &= e^{n(h(y)-h(x))} \left[\frac{-e^{ni(\text{Im } \lambda_{1+}(y)-\text{Im } \lambda_{1+}(x))} + e^{-ni(\text{Im } \lambda_{1+}(y)-\text{Im } \lambda_{1+}(x))}}{2\pi i(x - y)} + O(1) \right] \\ &= e^{n(h(y)-h(x))} \left[\frac{\sin(n\text{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)))}{\pi(x - y)} + O(1) \right] \end{aligned} \tag{9.2}$$

and the $O(1)$ holds uniformly in n . Letting $y \rightarrow x$ and noting that by (3.14) and (3.9),

$$\frac{d}{dy} \operatorname{Im} \lambda_{1+}(y) = \operatorname{Im} \xi_{1+}(y) = \pi \rho(y), \tag{9.3}$$

we obtain by l'Hopital's rule,

$$K_n(x, x) = n\rho(x) + O(1), \tag{9.4}$$

which proves Theorem 1.1 if $x \in (z_2, z_1)$. The proof for $x \in (-z_1, -z_2)$ is similar, and also follows because of symmetry.

For $x \in (-\infty, -z_1) \cup (-z_2, z_2) \cup (z_1, \infty)$, we have that $K_n(x, x)$ decreases exponentially fast. For example, for $x > z_1$, we have that

$$K_n(x, x) = O\left(e^{-n(\lambda_1(x) - \lambda_2(x))}\right) \quad \text{as } n \rightarrow \infty. \tag{9.5}$$

This follows from (4.9) and the observation that $T_+^{-1}(y)T_+(x) = I + O(x - y)$ as $y \rightarrow x$ if $x > z_1$. It is clear that (9.5) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} K_n(x, x) = 0. \tag{9.6}$$

We also have (9.6) if x is one of the edge points. In fact, for an edge point x it can be shown as in the proof of Theorem 1.3 that

$$\frac{1}{n} K_n(x, x) = O\left(\frac{1}{n^{1/3}}\right) \quad \text{as } n \rightarrow \infty. \tag{9.7}$$

This completes the proof of Theorem 1.1. \square

9.2. Proof of Theorem 1.2. We give the proof for $x_0 \in (z_2, z_1)$, the proof for $x_0 \in (-z_1, -z_2)$ being similar. We let

$$x = x_0 + \frac{u}{n\rho(x_0)}, \quad y = x_0 + \frac{v}{n\rho(x_0)}. \tag{9.8}$$

Then we have (9.2), and so by the definition (1.18) of \hat{K}_n ,

$$\frac{1}{n\rho(x_0)} \hat{K}_n(x, y) = \frac{\sin(n\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)))}{\pi(u - v)} + O\left(\frac{1}{n}\right). \tag{9.9}$$

Because of (9.3) we have by the mean value theorem,

$$\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)) = (x - y)\pi\rho(t) \tag{9.10}$$

for some t between x and y . Using (9.8) we get $t = x_0 + O(1/n)$ and

$$n\operatorname{Im}(\lambda_{1+}(x) - \lambda_{1+}(y)) = \pi(u - v) \frac{\rho(t)}{\rho(x_0)} = \pi(u - v) \left(1 + O\left(\frac{1}{n}\right)\right). \tag{9.11}$$

Inserting (9.11) into (9.9), we obtain

$$\frac{1}{n\rho(x_0)} \hat{K}_n(x, y) = \frac{\sin \pi(u - v)}{\pi(u - v)} + O\left(\frac{1}{n}\right) \tag{9.12}$$

which proves Theorem 1.2. \square

9.3. *Proof of Theorem 1.3.* We only give the proof of (1.22), since the proof of (1.23) is similar. We take ρ_1 as in (3.11) and we recall that $\beta'(z_1) = \rho_1^{2/3}$.

Take $u, v \in \mathbb{R}$ and let

$$x = z_1 + \frac{u}{(\rho_1 n)^{2/3}}, \quad y = z_1 + \frac{v}{(\rho_1 n)^{2/3}}. \quad (9.13)$$

Assume $u, v < 0$ so that we can use formula (5.14) for $K_n(x, y)$. Then we have that x belongs to $D(z_1, r)$, for n large enough, so that by (8.1), (7.7) and (7.14),

$$\begin{aligned} S_+(x) &= R(x)P_+(x) = R(x)\tilde{P}_+(x) \\ &= R(x)E_n(x)\Phi_+(n^{2/3}\beta(x))\text{diag}\left(e^{\frac{1}{2}n(\lambda_1-\lambda_2)_+(x)}, e^{-\frac{1}{2}n(\lambda_1-\lambda_2)_+(x)}, 1\right) \\ &= R(x)E_n(x)\Phi_+(n^{2/3}\beta(x))\text{diag}\left(e^{ni\text{Im}\lambda_{1+}(x)}, e^{-ni\text{Im}\lambda_{1+}(x)}, 1\right), \end{aligned} \quad (9.14)$$

and similarly for $S_+(y)$. Then we get from (5.14) and (1.18),

$$\begin{aligned} \frac{1}{(\rho_1 n)^{2/3}}\hat{K}_n(x, y) &= \frac{1}{2\pi i(u-v)}(-1 \ 1 \ 0)\Phi_+^{-1}(n^{2/3}\beta(y))E_n^{-1}(y)R^{-1}(y) \\ &\quad \times R(x)E_n(x)\Phi_+(n^{2/3}\beta(x))\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned} \quad (9.15)$$

Since $\rho^{2/3} = \beta'(z_1)$, we have as $n \rightarrow \infty$,

$$n^{2/3}\beta(x) = n^{2/3}\beta\left(z_1 + \frac{u}{(\rho_1 n)^{2/3}}\right) \rightarrow u, \quad (9.16)$$

which implies that $\Phi_+(n^{2/3}\beta(x)) \rightarrow \Phi_+(u)$. We use the second formula of (7.13) to evaluate $\Phi_+(u)$ (since $u < 0$), and it follows that

$$\lim_{n \rightarrow \infty} \Phi_+(n^{2/3}\beta(x))\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -y_1(u) - y_2(u) \\ -y'_1(u) - y'_2(u) \\ 0 \end{pmatrix} = \begin{pmatrix} y_0(u) \\ y'_0(u) \\ 0 \end{pmatrix}. \quad (9.17)$$

Similarly

$$\begin{aligned} \lim_{n \rightarrow \infty} (-1 \ 1 \ 0)\Phi_+^{-1}(n^{2/3}\beta(y)) &= (-1 \ 1 \ 0)\Phi_+^{-1}(v) \\ &= -2\pi i(-1 \ 1 \ 0)\begin{pmatrix} -y'_2(v) & y_2(v) & 0 \\ y'_1(v) & -y_1(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= -2\pi i(y'_2(v) + y'_1(v) - y_2(v) - y_1(v) \ 0) \\ &= -2\pi i(-y'_0(v) \ y_0(v) \ 0). \end{aligned} \quad (9.18)$$

The factor $-2\pi i$ comes from the inverse of $\Phi_+(v)$, since $\det \Phi = (-2\pi i)^{-1}$ by Wronskian relations.

Next, we recall that $R^{-1}(y)R(x) = I + O\left(\frac{x-y}{n}\right)$, so that by (9.13),

$$R^{-1}(y)R(x) = I + O\left(\frac{1}{n^{5/3}}\right). \quad (9.19)$$

The explicit form (7.15) for E_n readily gives

$$E_n(x) = O(n^{1/6}), \quad E_n^{-1}(y) = O(n^{1/6}), \quad E_n^{-1}(y)E_n(x) = I + O\left(\frac{1}{n^{1/3}}\right). \quad (9.20)$$

Combining (9.19) and (9.20), we have

$$\lim_{n \rightarrow \infty} E_n^{-1}(y)R^{-1}(y)R(x)E_n(x) = I. \quad (9.21)$$

Inserting (9.17), (9.18), and (9.21) into (9.15), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{cn^{2/3}} \hat{K}_n(x, y) &= \frac{1}{2\pi i(u-v)} \times (-2\pi i) \begin{pmatrix} -y'_0(v) & y_0(v) & 0 \end{pmatrix} \begin{pmatrix} y_0(u) \\ y'_0(u) \\ 0 \end{pmatrix} \\ &= \frac{y_0(u)y'_0(v) - y'_0(u)y_0(v)}{u-v}. \end{aligned} \quad (9.22)$$

Since $y_0 = \text{Ai}$, we have now completed the proof of (1.22) in case $u, v < 0$.

For the remaining cases where $u \geq 0$ and/or $v \geq 0$, we have to realize that we have not specified the rescaled kernel $\hat{K}_n(x, y)$ for x and/or y outside of $[-z_1, -z_2] \cup [z_2, z_1]$, since in (1.19) h is only defined there. We define

$$h(x) = -\frac{1}{4}x^2 + \frac{1}{2}(\lambda_1(x) + \lambda_2(x)), \quad x \in (z_1, \infty). \quad (9.23)$$

We will assume in the rest of the proof that $u > 0$ and $v > 0$. The case where u and v have opposite signs follows in a similar way: then we have to combine the calculations given below with the ones given above.

So let $u, v > 0$ and let x and y be as in (9.13). For the kernel K_n we start from the expression (4.9) in terms of T . Since $u > 0$, we have $x, y > z_1$, and so we have by (5.6), (8.1), (7.7) and (7.14),

$$\begin{aligned} T_+(x) &= S_+(x) = R(x)P_+(x) = R(x)\tilde{P}_+(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3(x) - \lambda_2(x))} \\ 0 & 0 & 1 \end{pmatrix} \\ &= R(x)E_n(x)\Phi_+(n^{2/3}\beta(x)) \\ &\quad \times \begin{pmatrix} e^{\frac{1}{2}n(\lambda_1(x) - \lambda_2(x))} & 0 & 0 \\ 0 & e^{-\frac{1}{2}n(\lambda_1(x) - \lambda_2(x))} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3(x) - \lambda_2(x))} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (9.24)$$

Then

$$T_+(x) \begin{pmatrix} e^{-n\lambda_1(x)} \\ 0 \\ 0 \end{pmatrix} = e^{-\frac{1}{2}n(\lambda_1(x) - \lambda_2(x))} R(x)E_n(x)\Phi_+(n^{2/3}\beta(x)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (9.25)$$

As before, we have $\Phi_+(n^{2/3}\beta(x)) \rightarrow \Phi_+(u)$ as $n \rightarrow \infty$. Now we use the first formula of (7.13) to evaluate $\Phi_+(u)$ so that

$$\lim_{n \rightarrow \infty} \Phi_+(n^{2/3}\beta(x)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Phi_+(u) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_0(u) \\ y'_0(u) \\ 0 \end{pmatrix}. \quad (9.26)$$

We have as in (9.24)

$$T_+^{-1}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_3(y)-\lambda_2(y))} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}n(\lambda_1(y)-\lambda_2(y))} & 0 & 0 \\ 0 & e^{\frac{1}{2}n(\lambda_1(y)-\lambda_2(y))} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \times \Phi_+^{-1}(n^{2/3}\beta(y))E_n^{-1}(y)R^{-1}(y), \quad (9.27)$$

so that

$$(0 \ e^{n\lambda_2(y)} \ e^{n\lambda_3(y)}) T_+^{-1}(y) = e^{\frac{1}{2}n(\lambda_1(y)-\lambda_2(y))} (0 \ 1 \ 0) \Phi_+^{-1}(n^{2/3}\beta(y))E_n^{-1}(y)R^{-1}(y). \quad (9.28)$$

We have

$$\lim_{n \rightarrow \infty} (0 \ 1 \ 0) \Phi_+^{-1}(n^{2/3}\beta(y)) = (0 \ 1 \ 0) \Phi_+^{-1}(u) \\ = (-2\pi i) (-y'_0(u) \ y_0(u) \ 0), \quad (9.29)$$

and as before we have (9.21).

Inserting (9.25) and (9.28) into (9.20) and using the limits (9.21), (9.26) and (9.29), we arrive at (1.22) in the case $u, v > 0$.

This completes the proof of Theorem 1.3. \square

10. Large n Asymptotics of the Multiple Hermite Polynomials

As noted in Sect. 2, see also [7], we have that the $(1, 1)$ entry of the solution Y of the RH problem (1.4)–(1.5) is a monic polynomial P_n of degree n satisfying

$$\int_{-\infty}^{\infty} P_n(x)x^k w_j(x)dx = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2.$$

For the case $w_1(x) = e^{-n(\frac{1}{2}x^2-ax)}$, $w_2(x) = e^{-n(\frac{1}{2}x^2+ax)}$, this polynomial is called a multiple Hermite polynomial [3, 8, 27]. The asymptotic analysis of the RH problem done in Sects. 4–9, also yields the strong asymptotics of the multiple Hermite polynomials (as $n \rightarrow \infty$ with n even and $n_1 = n_2$) in every part of the complex plane. We describe these asymptotics here. Recall that P_n is the average characteristic polynomial of the random matrix ensemble (1.1), see (1.8).

We will partition the complex plane into 3 regions:

- (1) Outside of the lenses and of the disks $D(\pm z_j, r)$, $j = 1, 2$.
- (2) Inside of the lenses but outside of the disks.
- (3) Inside of the disks.

We will derive the large n asymptotics of the multiple Hermite polynomials in these 3 regions.

(1) *Region outside of the lenses and of the disks.* In this region, we have by (5.6) and (8.1),

$$T(z) = R(z)M(z), \tag{10.1}$$

hence by (4.1)

$$\begin{aligned} & \text{diag}(e^{-nl_1}, e^{-nl_2}, e^{-nl_3})Y(z)\text{diag}(e^{-\frac{n}{2}z^2}, e^{-naz}, e^{naz}) \\ &= R(z)M(z)\text{diag}(e^{-n\lambda_1(z)}, e^{-n\lambda_2(z)}, e^{-n\lambda_3(z)}). \end{aligned} \tag{10.2}$$

By restricting this matrix equation to the element (1, 1) we obtain that

$$P_n(z)e^{-\frac{n}{2}z^2} = e^{-n\lambda(z)} \sum_{j=1}^3 R_{1j}(z)M_{j1}(z), \tag{10.3}$$

where

$$\lambda(z) \equiv \lambda_1(z) - l_1 = \int^z \xi_1(s) ds, \tag{10.4}$$

and as $z \rightarrow \infty$,

$$\lambda(z) = \frac{z^2}{2} - \ln z + O(z^{-2}). \tag{10.5}$$

In the sum over j in (10.3) the term $j = 1$ dominates and we obtain because of (6.16) that

$$P_n(z)e^{-\frac{n}{2}z^2} = \frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} e^{-n\lambda(z)} \left(1 + O\left(\frac{1}{n(|z| + 1)}\right) \right), \tag{10.6}$$

where for the square root we use the principal branch (the one that is positive for $z > z_1$), with two cuts, $[-z_1, -z_2]$ and $[z_2, z_1]$.

(2) *Region inside of the lenses but outside of the disks.* In this region, we get from (5.2), (5.4) and (8.1),

$$T(z) = R(z)M(z)L(z)^{-1}, \tag{10.7}$$

where $L(z)$ is the matrix on the right in (5.2) and (5.4). Hence by (4.1),

$$\begin{aligned} & \text{diag}(e^{-nl_1}, e^{-nl_2}, e^{-nl_3})Y(z)\text{diag}(e^{-\frac{n}{2}z^2}, e^{-naz}, e^{naz}) \\ &= R(z)M(z)L(z)^{-1}\text{diag}(e^{-n\lambda_1(z)}, e^{-n\lambda_2(z)}, e^{-n\lambda_3(z)}). \end{aligned} \tag{10.8}$$

Consider z the upper lens region on $[z_2, z_1]$. Then

$$L(z) = \begin{pmatrix} 1 & 0 & 0 \\ -e^{n(\lambda_1(z) - \lambda_2(z))} & 1 & -e^{n(\lambda_3(z) - \lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix}, \tag{10.9}$$

hence

$$L(z)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ e^{n(\lambda_1(z)-\lambda_2(z))} & 1 & e^{n(\lambda_3(z)-\lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.10)$$

and the first column of the matrix $M(z)L(z)^{-1}\text{diag}(e^{-n\lambda_1(z)}, e^{-n\lambda_2(z)}, e^{-n\lambda_3(z)})$ is

$$\begin{pmatrix} M_1(\xi_1(z))e^{-n\lambda_1(z)} + M_1(\xi_2(z))e^{-n\lambda_2(z)} \\ M_2(\xi_1(z))e^{-n\lambda_1(z)} + M_2(\xi_2(z))e^{-n\lambda_2(z)} \\ M_3(\xi_1(z))e^{-n\lambda_1(z)} + M_3(\xi_2(z))e^{-n\lambda_2(z)} \end{pmatrix}, \quad (10.11)$$

see (6.6). By restricting Eq. (10.8) to the (1, 1) entry, and using (6.16) and (8.6), we obtain that in the upper lens region on $[z_2, z_1]$,

$$\begin{aligned} P_n(z)e^{-\frac{n}{2}z^2} &= \left[\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O\left(\frac{1}{n}\right) \right] e^{-n\lambda_1(z)+nl_1} \\ &+ \left[\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O\left(\frac{1}{n}\right) \right] e^{-n\lambda_2(z)+nl_1}, \end{aligned} \quad (10.12)$$

where

$$\lambda_k(z) = \int_{z_1}^z \xi_k(s) ds, \quad k = 1, 2. \quad (10.13)$$

In the same way we obtain that in the lower lens region on $[z_2, z_1]$,

$$\begin{aligned} P_n(z)e^{-\frac{n}{2}z^2} &= \left[\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} + O\left(\frac{1}{n}\right) \right] e^{-n\lambda_1(z)+nl_1} \\ &- \left[\frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} + O\left(\frac{1}{n}\right) \right] e^{-n\lambda_2(z)+nl_1}. \end{aligned} \quad (10.14)$$

For $z = x$ real, $x \in [z_2 + r, z_1 - r]$, both (10.12) and (10.14) can be rewritten in the form

$$P_n(x)e^{-\frac{n}{2}x^2} = \left\{ A(x) \cos[n \text{Im } \lambda_{1+}(x) - \varphi(x)] + O\left(\frac{1}{n}\right) \right\} e^{-n \text{Re } \lambda_{1+}(x) + nl_1}, \quad (10.15)$$

where

$$A(x) = 2 \left| \frac{\xi_{1+}^2(x) - a^2}{\sqrt{(\xi_{1+}^2(x) - p^2)(\xi_{1+}^2(x) - q^2)}} \right| \quad (10.16)$$

and

$$\varphi(x) = \arg \frac{\xi_{1+}^2(x) - a^2}{\sqrt{(\xi_{1+}^2(x) - p^2)(\xi_{1+}^2(x) - q^2)}}. \quad (10.17)$$

By using Eq. (3.9), we can also rewrite (10.15) in terms of the eigenvalue density function $\rho(x)$,

$$P_n(x)e^{-\frac{n}{2}x^2} = \left\{ A(x) \cos \left[n\pi \int_{z_1}^x \rho(s) ds - \varphi(x) \right] + O\left(\frac{1}{n}\right) \right\} e^{-n \operatorname{Re} \lambda_{1+(x)+nl_1}. \quad (10.18)$$

Equation (10.18) clearly displays the oscillating behavior of P_n on the interval $[z_2 + r, z_1 - r]$. It also shows that the zeros of $P_n(x)$ are asymptotically distributed like $\rho(x) dx$, the limiting probability distribution of eigenvalues. Similar formulae can be derived on the interval $[-z_1 + r, -z_2 - r]$.

(3) *Region inside of the disks..* Consider the disk $D(z_1, r)$. In the regions I and IV, we have by (7.7), (8.1) and (8.6),

$$T(z) = R(z)P(z) = \left(I + O(n^{-1}) \right) \tilde{P}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_3(z) - \lambda_2(z))} \\ 0 & 0 & 1 \end{pmatrix}, \quad (10.19)$$

hence by (4.1), (7.14), and (7.15),

$$\begin{aligned} & \operatorname{diag}(e^{-nl_1}, e^{-nl_2}, e^{-nl_3})Y(z)\operatorname{diag}(e^{-\frac{n}{2}z^2}, e^{-naz}, e^{naz}) \\ &= (I + O(n^{-1}))\sqrt{\pi}M(z) \begin{pmatrix} n^{1/6}\beta(z)^{1/4} & -n^{-1/6}\beta(z)^{-1/4} & 0 \\ -in^{1/6}\beta(z)^{1/4} & -in^{-1/6}\beta(z)^{-1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ & \quad \times \Phi(n^{2/3}\beta(z))\operatorname{diag}(e^{-n\alpha(z)}, e^{-n\alpha(z)}, e^{-n\lambda_3(z)}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \quad (10.20)$$

where

$$\alpha(z) = \frac{\lambda_1(z) + \lambda_2(z)}{2}. \quad (10.21)$$

By restricting Eq. (10.20) to the (1, 1) entry, and using the first expression of (7.13) (in region I) or the fourth expression of (7.13) (in region IV) to evaluate $\Phi(n^{2/3}\beta(z))$, and (6.16) to evaluate $M(z)$, we obtain that

$$\begin{aligned} P_n(z)e^{-\frac{n}{2}z^2} &= \sqrt{\pi} \left[n^{1/6}B(z)\operatorname{Ai}(n^{2/3}\beta(z))(1 + O(n^{-1})) \right. \\ & \quad \left. + n^{-1/6}C(z)\operatorname{Ai}'(n^{2/3}\beta(z))(1 + O(n^{-1})) \right] e^{-n\alpha(z)+nl_1}, \end{aligned} \quad (10.22)$$

where

$$B(z) = \beta(z)^{1/4} \left(\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i \frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} \right) \quad (10.23)$$

and

$$C(z) = \beta(z)^{-1/4} \left(-\frac{\xi_1^2(z) - a^2}{\sqrt{(\xi_1^2(z) - p^2)(\xi_1^2(z) - q^2)}} - i \frac{\xi_2^2(z) - a^2}{\sqrt{(\xi_2^2(z) - p^2)(\xi_2^2(z) - q^2)}} \right). \quad (10.24)$$

The same asymptotics, (10.22), holds in regions II and III as well. Thus, (10.22) holds in the full disk $D(z_1, r)$. It may be verified that the functions $B(z)$ and $C(z)$ are analytic in $D(z_1, r)$.

This approach allows one to derive a formula similar to (10.22) in all the other disks $D(\pm z_j, r)$ as well.

Appendix A. Recurrence Equations for Multiple Hermite Polynomials

From orthogonality equation (2.2), we obtain that as $z \rightarrow \infty$,

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{P_{n_1, n_2}(u) w_k(u)}{u - z} du &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} P_{n_1, n_2}(u) w_k(u) \left(\frac{1}{z} + \frac{u}{z^2} + \dots \right) du \\ &= -\frac{1}{2\pi i} \left(\frac{h_{n_1, n_2}^{(k)}}{z^{n_k+1}} + \frac{q_{n_1, n_2}^{(k)}}{z^{n_k+2}} + \dots \right), \quad k = 1, 2, \end{aligned} \quad (A.1)$$

where for $k = 1, 2$, $h_{n_1, n_2}^{(k)}$ is defined in (2.3) and

$$q_{n_1, n_2}^{(k)} = \int_{-\infty}^{\infty} P_{n_1, n_2}(x) x^{n_k+1} w_k(x) dx. \quad (A.2)$$

This implies that

$$\Psi_{n_1, n_2}(z) = \left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right) \text{diag} \left(z^n e^{-\frac{1}{2} N z^2}, c_1^{-1} z^{-n_1} e^{-N a z}, c_2^{-1} z^{-n_2} e^{N a z} \right), \quad (A.3)$$

where

$$\Psi_{n_1, n_2}^{(1)} = \begin{pmatrix} P_{n_1, n_2} & \frac{h_{n_1, n_2}^{(1)}}{h_{n_1-1, n_2}^{(1)}} & \frac{h_{n_1, n_2}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \\ 1 & \frac{q_{n_1-1, n_2}^{(1)}}{h_{n_1-1, n_2}^{(1)}} & \frac{h_{n_1-1, n_2}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \\ 1 & \frac{h_{n_1, n_2-1}^{(1)}}{h_{n_1-1, n_2}^{(1)}} & \frac{q_{n_1, n_2-1}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \end{pmatrix}, \quad (A.4)$$

and $P_{n_1, n_2}(z) = z^n + p_{n_1, n_2} z^{n-1} + \dots$. Set

$$U_{n_1, n_2}(z) = \Psi_{n_1+1, n_2}(z) \Psi_{n_1, n_2}(z)^{-1}. \quad (A.5)$$

Then by (2.9), $U_{n_1, n_2+}(x) = U_{n_1, n_2-}(x)$ (i.e., no jump on the real line) and as $z \rightarrow \infty$,

$$\begin{aligned} U_{n_1, n_2}(z) &\cong \left(I + \frac{\Psi_{n_1+1, n_2}^{(1)}}{z} + \dots \right) \begin{pmatrix} z & 0 & 0 \\ 0 & z^{-1} \frac{h_{n_1, n_2}^{(1)}}{h_{n_1-1, n_2}^{(1)}} & 0 \\ 0 & 0 & \frac{h_{n_1+1, n_2-1}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \end{pmatrix} \left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right)^{-1} \\ &= zP_1 + \Psi_{n_1+1, n_2}^{(1)} P_1 - P_1 \Psi_{n_1, n_2}^{(1)} + \frac{h_{n_1+1, n_2-1}^{(2)}}{h_{n_1, n_2-1}^{(2)}} P_3 + O\left(\frac{1}{z}\right), \end{aligned} \quad (\text{A.6})$$

where

$$P_1 = \text{diag}(1, 0, 0), \quad P_2 = \text{diag}(0, 1, 0), \quad P_3 = \text{diag}(0, 0, 1). \quad (\text{A.7})$$

Since $U_{n_1, n_2}(z)$ is analytic on the complex plane, Eq. (A.6) implies, by the Liouville theorem, that

$$\begin{aligned} U_{n_1, n_2}(z) &= zP_1 + \Psi_{n_1+1, n_2}^{(1)} P_1 - P_1 \Psi_{n_1, n_2}^{(1)} + \frac{h_{n_1+1, n_2-1}^{(2)}}{h_{n_1, n_2-1}^{(2)}} P_3 \\ &= \begin{pmatrix} z - b_{n_1, n_2} & -c_{n_1, n_2} & -d_{n_1, n_2} \\ 1 & 0 & 0 \\ 1 & 0 & e_{n_1, n_2} \end{pmatrix}, \end{aligned} \quad (\text{A.8})$$

where

$$c_{n_1, n_2} = \frac{h_{n_1, n_2}^{(1)}}{h_{n_1-1, n_2}^{(1)}} \neq 0, \quad d_{n_1, n_2} = \frac{h_{n_1, n_2}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \neq 0, \quad e_{n_1, n_2} = \frac{h_{n_1+1, n_2-1}^{(2)}}{h_{n_1, n_2-1}^{(2)}} \neq 0. \quad (\text{A.9})$$

Thus, we obtain the matrix recurrence equation,

$$\Psi_{n_1+1, n_2}(z) = U_{n_1, n_2}(z) \Psi_{n_1, n_2}(z). \quad (\text{A.10})$$

By restricting it to the element (1, 1) we obtain that

$$P_{n_1+1, n_2}(z) = (z - b_{n_1, n_2}) P_{n_1, n_2}(z) - c_{n_1, n_2} P_{n_1-1, n_2}(z) - d_{n_1, n_2} P_{n_1, n_2-1}(z), \quad (\text{A.11})$$

and by restricting it to the element (3, 1) we obtain that

$$P_{n_1+1, n_2-1}(z) = P_{n_1, n_2}(z) + e_{n_1, n_2} P_{n_1, n_2-1}(z). \quad (\text{A.12})$$

Similar to (A.10), we have another recurrence equation,

$$\Psi_{n_1, n_2+1}(z) = \tilde{U}_{n_1, n_2}(z) \Psi_{n_1, n_2}(z), \quad (\text{A.13})$$

where

$$\tilde{U}_{n_1, n_2}(z) = \begin{pmatrix} z - \tilde{b}_{n_1, n_2} & -c_{n_1, n_2} & -d_{n_1, n_2} \\ 1 & \tilde{e}_{n_1, n_2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (\text{A.14})$$

and

$$\tilde{e}_{n_1, n_2} = \frac{h_{n_1-1, n_2+1}^{(1)}}{h_{n_1-1, n_2}^{(1)}} \neq 0. \quad (\text{A.15})$$

By restricting (A.13) to the elements (1, 1) and (2, 1), we obtain the equations,

$$P_{n_1, n_2+1}(z) = (z - \tilde{b}_{n_1, n_2})P_{n_1, n_2}(z) - c_{n_1, n_2}P_{n_1-1, n_2}(z) - d_{n_1, n_2}P_{n_1, n_2-1}(z), \quad (\text{A.16})$$

and

$$P_{n_1-1, n_2+1}(z) = P_{n_1, n_2}(z) + \tilde{e}_{n_1, n_2}P_{n_1-1, n_2}(z). \quad (\text{A.17})$$

Appendix B. Differential Equations for Multiple Hermite Polynomials

Set

$$A_{n_1, n_2}(z) = \frac{1}{N} \Psi'_{n_1, n_2}(z) \Psi_{n_1, n_2}(z)^{-1}. \quad (\text{B.1})$$

It follows from (2.9), that $A_{n_1, n_2}(z)$ has no jump on the real axis, so that it is analytic on the complex plane. By differentiating (A.3) we obtain that as $z \rightarrow \infty$,

$$A_{n_1, n_2}(z) = \left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right) \begin{pmatrix} -z & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right)^{-1} + \mathcal{O}\left(\frac{1}{z}\right). \quad (\text{B.2})$$

Since $A_{n_1, n_2}(z)$ is analytic, we obtain that

$$A_{n_1, n_2}(z) = - \left[\left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right) \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(I + \frac{\Psi_{n_1, n_2}^{(1)}}{z} + \dots \right)^{-1} \right]_{\text{pol}} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad (\text{B.3})$$

where $[f(z)]_{\text{pol}}$ means the polynomial part of $f(z)$ at infinity. From (B.1) we get the differential equation,

$$\Psi'_{n_1, n_2}(z) = N A_{n_1, n_2}(z) \Psi_{n_1, n_2}(z), \quad (\text{B.4})$$

and (B.3) reduces to

$$A_{n_1, n_2}(z) = \begin{pmatrix} -z & c_{n_1, n_2} & d_{n_1, n_2} \\ -1 & -a & 0 \\ -1 & 0 & a \end{pmatrix}. \quad (\text{B.5})$$

Appendix C. Proof of Proposition 2.1

Equations (A.10), (A.13), (B.4) form a Lax pair for multiple Hermite polynomials. Their compatibility conditions are

$$\begin{aligned}\frac{1}{N}U'_{n_1, n_2}(z) &= A_{n_1+1, n_2}(z)U_{n_1, n_2}(z) - U_{n_1, n_2}(z)A_{n_1, n_2}(z), \\ \frac{1}{N}\tilde{U}'_{n_1, n_2}(z) &= A_{n_1, n_2+1}(z)\tilde{U}_{n_1, n_2}(z) - \tilde{U}_{n_1, n_2}(z)A_{n_1, n_2}(z).\end{aligned}\quad (\text{C.1})$$

This gives the equations,

$$\begin{aligned}b_{n_1, n_2} &= a, & c_{n_1+1, n_2} &= c_{n_1, n_2} + \frac{1}{N}, & d_{n_1+1, n_2} &= d_{n_1, n_2}, & e_{n_1, n_2} &= -2a, \\ \tilde{b}_{n_1, n_2} &= -a, & c_{n_1, n_2+1} &= c_{n_1, n_2}, & d_{n_1, n_2+1} &= d_{n_1, n_2} + \frac{1}{N}, & \tilde{e}_{n_1, n_2} &= 2a.\end{aligned}\quad (\text{C.2})$$

Since $c_{0, n_2} = d_{n_1, 0} = 0$, we obtain that

$$c_{n_1, n_2} = \frac{n_1}{N}, \quad d_{n_1, n_2} = \frac{n_2}{N}.\quad (\text{C.3})$$

This proves the first equation in (2.10) and Eq. (2.11). Similarly we obtain that $\tilde{e}_{n_1, n_2} = 2a$ and this proves the second equation in (2.10). Proposition 2.1 is proved. \square

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