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# Double scaling limit in random matrix models and a nonlinear hierarchy of differential equations

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## Abstract

We study the critical behaviour of a random Hermitian one-matrix model with nonsymmetric interaction at a critical point, in which the eigenvalue density function has a zero of degree  $2m$ ,  $m \geq 1$ , inside a cut. We prove that in the generic case,  $m = 1$ , the model exhibits a third-order phase transition in temperature. We formulate an ansatz for the double scaling limit of recurrence coefficients, which is consistent with the quasiperiodic asymptotics of recurrence coefficients in the low temperature region, and from this ansatz we derive the Painlevé II hierarchy of ordinary differential equations for the recurrence coefficients. In addition, we derive an integral kernel which governs the double scaling limit of correlation functions.

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## 1. Introduction

We consider the unitary ensemble of random matrices,

$$d\mu_N(M) = Z_N^{-1} \exp(-N \operatorname{Tr} V(M)) dM \quad Z_N = \int_{\mathcal{H}_N} \exp(-N \operatorname{Tr} V(M)) dM \quad (1.1)$$

on the space  $\mathcal{H}_N$  of Hermitian  $N \times N$  matrices  $M = (M_{ij})_{1 \leq i, j \leq N}$ , where  $V(x)$  is a polynomial,  $V(x) = v_p x^p + v_{p-1} x^{p-1} + \dots$ , of an even degree  $p$  with  $v_p > 0$ . In this paper we are concerned with the critical behaviour of the polynomial  $V(x)$  such that the corresponding equilibrium measure is supported by the segment  $[-2, 2]$ , with a density function of the form

$$\rho(x) = Z^{-1} (x - c)^{2m} \sqrt{4 - x^2} \quad Z = \int_{\mathbb{R}} (x - c)^{2m} \sqrt{4 - x^2} dx \quad (1.2)$$

where  $-2 < c < 2$  and  $m = 1, 2, \dots$ . The choice of the support segment is obviously not important, because by a shift and a dilation one can reduce any segment to  $[-2, 2]$ . The parameter  $m$  determines the degree of degeneracy of the equilibrium measure at  $x = c$ . Observe that when  $c \neq 0$ , density (1.2) is not symmetric.

Our results are summarized as follows. We are interested in two problems:

- (1) The singularity of the infinite volume free energy at the critical point.
- (2) The double scaling limit, i.e. the limit of rescaled correlation functions as simultaneously the volume goes to infinity and the parameter  $t$  goes to  $t^c$ , with an appropriate relation between  $t - t^c$  and the volume.

*Case  $m = 1$ . Free energy.* We will evaluate the derivatives in  $T$  of the (infinite volume) free energy  $F(T)$ , where  $T > 0$  is the temperature, and we will show that  $F(T)$  can be analytically continued through the critical value  $T_c$  both from below and from above  $T_c$ . In addition, we will show that  $F(T)$  and its first two derivatives are continuous at  $T = T_c$ , while the third derivative has a jump. This will prove that at  $T = T_c$  the phase transition is of the third order, giving an extension of the result of [GW] where the third-order phase transition was shown for the case of a *symmetric* critical  $V(x)$  in the *circular* ensemble of random matrices.

*Case  $m = 1$ . Double scaling limit.* The key problem here is to obtain an asymptotic formula for the recurrence coefficients of the corresponding orthogonal polynomials. In the case under consideration, the double scaling limit describes a transition from a fixed point behaviour of the recurrence coefficients at  $T > T_c$  to a quasiperiodic behaviour at  $T < T_c$  ([DKMVZ]; see also [BDE]), and the problem is to derive an asymptotic formula for the recurrence coefficients in the transition region. We will formulate a double scaling limit ansatz for the recurrence coefficients (see formulae (2.69)–(2.71)), which is consistent with the quasiperiodic asymptotics in the low temperature region. As was shown in [DKMVZ] (see also [BDE]), the quasiperiodic asymptotics at  $T < T_c$  is expressed in terms of the Jacobi elliptic theta function. In the double scaling limit one of the periods of the theta function approaches zero, and the theta function degenerates into a trigonometric function. This explains why formulae (2.69)–(2.71) involve trigonometric functions. We will show that ansatz (2.69)–(2.71) leads to the critical, Hastings–McLeod solution to the Painlevé II differential equation. We will show then that the double scaling limit of the correlation functions is governed by an integral kernel, which is constructed from solutions of the linear  $2 \times 2$  system associated with the Hastings–McLeod solution (cf [BI2]).

In the symmetric case ( $c = 0$ ), the recurrence coefficients in the low temperature region have a period 2 asymptotics. In this case the double scaling ansatz for recurrence coefficients is more obvious. The Painlevé II equation was derived in the symmetric case in [DSS]. A detailed analysis of the phase diagram for symmetric  $V(M)$  of degrees 4 and 6 was given in [DDJT], together with concrete calculations of the double scaling limit of the orders  $m = 1$  and 2. For the circular ensemble, the whole hierarchy of the double scaling limits was obtained in the symmetric case in [PeS]. It is also worth mentioning earlier physical works [BKa, GM, DS] which are concerned with the double scaling limit of the Painlevé I type.

A rigorous proof of the double scaling asymptotics is very difficult, even in the symmetric case, and was given in [BI2] (unitary ensemble, quartic  $V(M)$ ) and in [BDJ] (circular ensemble,  $V(M) = a(M + M^{-1})$ ). Both [BI2, BDJ] are based on the Riemann–Hilbert approach to orthogonal polynomials, developed in [FIK, BI1, DKMVZ].

The double scaling limit for nonsymmetric  $V(M)$  was considered in [HMPN] (see also references therein), under the assumption that the recurrence coefficients have period 2 (see formula (2.6) in [HMPN]). As a matter of fact, in the case under consideration, as  $T \rightarrow T_c - 0$ , the frequency of the quasiperiodic recurrence coefficients approaches  $\varepsilon = \pi^{-1} \arccos \frac{c}{2}$ , which

is equal to  $\frac{1}{2}$ , the period 2 frequency, only in the symmetric case,  $c = 0$ . It is also worth mentioning, that in the two-cut region, the free energy of the quartic matrix model, with  $V(M) = \frac{g}{4}M^4 + \frac{t}{2}M^2 + hM$ , is analytic in the external field  $h$  at  $h = 0$ , and therefore there is no symmetry-breaking phenomenon at  $h = 0$ , assumed in [BDJT].

*General case,  $m \geq 1$ . Double scaling limit.* We will formulate a double scaling ansatz for any  $m \geq 1$ , and we will derive from this ansatz a hierarchy of nonlinear ordinary differential equations which give the double scaling limit of the recurrence coefficients for all  $m \geq 1$ . The hierarchy admits a Lax pair of linear differential equations and it can be constructed in the framework of the general theory of isomonodromic deformations [IN]. Our particular hierarchy is known as the Painlevé II hierarchy [Kit] and it is related to self-similar solutions of the mKdV equation [PeS] (see also [Moo, HMPN]). For symmetric  $V$ , the case  $m = 1, 2$  was considered in [DDJT], for the Hermitian random matrices, and the general case,  $m \geq 1$ , was considered in [PeS], for the circular ensemble of random matrices.

Let us briefly recall some general formulae for the unitary ensemble of random matrices. The distribution of eigenvalues  $\lambda = \{\lambda_j, j = 1, \dots, N\}$  of  $M$  in ensemble (1.1) is given by the Weyl formula (see, e.g., [Meh, TW]),

$$d\mu_N(\lambda) = \tilde{Z}_N^{-1} \exp(-N H_N(\lambda)) d\lambda \quad \tilde{Z}_N = \int_{\Lambda_N} \exp(-N H_N(\lambda)) d\lambda \tag{1.3}$$

where  $\Lambda_N$  is the symmetrized  $\mathbb{R}^N$ ,  $\Lambda_N = \mathbb{R}^N/S(N)$ , and

$$H_N(\lambda) = -\frac{2}{N} \sum_{1 \leq j < k \leq N} \log |\lambda_j - \lambda_k| + \sum_{j=1}^N V(\lambda_j). \tag{1.4}$$

Let  $dv_N(x) = \rho_N(x) dx$  be the distribution of the eigenvalues on the line, so that for any test function  $\varphi(x) \in C_0^\infty$ ,

$$\int_{\Lambda_N} \left[ \frac{1}{N} \sum_{j=1}^N \varphi(\lambda_j) \right] d\mu_N(\lambda) = \int_{-\infty}^{\infty} \varphi(x) dv_N(x). \tag{1.5}$$

As  $N \rightarrow \infty$ , there exists a weak limit of  $dv_N(x)$ ,

$$dv_\infty(x) = \lim_{N \rightarrow \infty} dv_N(x). \tag{1.6}$$

To determine the limit (cf [BIPZ, DGZ] and others), consider the energy functional on the space of probability measures on the line,

$$I(dv(x)) = - \iint_{\mathbb{R}^2} \log |x - y| dv(x) dv(y) + \int_{\mathbb{R}} V(x) dv(y). \tag{1.7}$$

Then  $H_N(\lambda)$  in (1.4) can be written as

$$H_N(\lambda) = NI(dv(x; \lambda)) \tag{1.8}$$

where  $dv(x; \lambda)$  is a discrete probability measure with atoms at  $\lambda_j$ ,

$$dv(x; \lambda) = \frac{1}{N} \sum_{j=1}^N \delta(x - \lambda_j) dx. \tag{1.9}$$

Hence,

$$d\mu_N(\lambda) = \tilde{Z}_N^{-1} \exp(-N^2 I(dv(x; \lambda))) d\lambda. \tag{1.10}$$

Because of the factor  $N^2$  in the exponent, one can expect that as  $N \rightarrow \infty$ , the measures  $d\mu_N(\lambda)$  are localized in a shrinking vicinity of an *equilibrium measure*  $dv_{\text{eq}}(x)$ , which minimizes the

functional  $I(d\nu(x))$ , and therefore, one expects the limit (1.6) to exist with  $d\nu_\infty(x) = d\nu_{\text{eq}}(x)$ . A rigorous proof of the existence and uniqueness of the equilibrium measure, its properties, and the existence of limit (1.6) with  $d\nu_\infty(x) = d\nu_{\text{eq}}(x)$ , was given in [BPS, Joh].

The equilibrium measure  $d\nu_{\text{eq}}(x)$  is supported by a finite number of segments  $[a_j, b_j]$ ,  $j = 1, \dots, q$ , and it is absolutely continuous with respect to the Lebesgue measure,  $d\nu_{\text{eq}}(x) = \rho(x) dx$ , with a density function  $\rho(x)$  of the form

$$\rho(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x) \quad R(x) = \prod_{j=1}^q (x - a_j)(x - b_j) \quad (1.11)$$

where  $h(x)$  is a polynomial of degree,  $\deg h = p - q - 1$ , and  $R_+^{1/2}(x)$  means the value on the upper cut of the principal sheet of the function  $R^{1/2}(z)$  with cuts on  $J$ . The equilibrium measure is uniquely determined by the Euler–Lagrange conditions (see [DKMVZ]): for some real constant  $l$ ,

$$2 \int_{\mathbb{R}} \log |x - s| d\nu_{\text{eq}}(s) - V(x) = l \quad \text{for } x \in \bigcup_{j=1}^q [a_j, b_j] \quad (1.12)$$

$$2 \int_{\mathbb{R}} \log |x - s| d\nu_{\text{eq}}(s) - V(x) \leq l \quad \text{for } x \in \mathbb{R} \setminus \bigcup_{j=1}^q [a_j, b_j]. \quad (1.13)$$

Equations (1.11) and (1.12) imply that

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2} \quad (1.14)$$

where

$$\omega(z) \equiv \int_J \frac{\rho(x) dx}{z - x} = z^{-1} + O(z^{-2}) \quad z \rightarrow \infty. \quad (1.15)$$

In addition, (1.12) implies that

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2} dx = 0 \quad j = 1, \dots, q - 1 \quad (1.16)$$

which shows that  $h(x)$  has at least one zero on each interval  $b_j < x < a_{j+1}$ ;  $j = 1, \dots, q - 1$ . From (1.14) we obtain that

$$V'(z) = \text{Pol}[h(z)R^{1/2}(z)] \quad \text{Res}_{z=\infty}[h(z)R^{1/2}(z)] = -2 \quad (1.17)$$

and

$$h(z) = \text{Pol} \left[ \frac{V'(z)}{R^{1/2}(z)} \right] \quad (1.18)$$

where  $\text{Pol}[f(z)]$  is the polynomial part of  $f(z)$  at  $z = \infty$ . The latter equation expresses  $h(z)$  in terms of  $V(z)$  and the endpoints,  $a_1, b_1, \dots, a_q, b_q$ . The endpoints can be further found from (1.17), which gives  $q + 1$  equation on  $a_1, \dots, b_q$ , and from (1.16), which gives the remaining  $q - 1$  equations.

The equilibrium measure  $d\nu_{\text{eq}}(x)$  is called *regular* (otherwise *singular*), see [DKMVZ], if

$$h(x) \neq 0 \quad \text{for } x \in \bigcup_{j=1}^q [a_j, b_j] \quad (1.19)$$

and

$$2 \int \log |x - s| \, d\nu_{\text{eq}}(s) - V(x) < l \quad \text{for } x \in \mathbb{R} \setminus \bigcup_{j=1}^q [a_j, b_j]. \quad (1.20)$$

The polynomial  $V(x)$  is called *critical* if the corresponding equilibrium measure  $d\nu_{\text{eq}}(x)$  is singular. To study the critical behaviour in the vicinity of a critical polynomial  $V(x)$ , one embeds  $V(x)$  into a parametric family  $V(x; t), t = (t_1, \dots, t_r)$ , so that for some  $t^c, V(x; t^c) = V(x)$ , and the problem is then to evaluate the asymptotics of eigenvalue correlation functions as  $t \rightarrow t^c$ . The number of parameters  $r$  depends, in general, on the degree of degeneracy of the equilibrium measure  $d\nu_{\text{eq}}(x)$ . Let us consider first the critical behaviour for the nonsymmetric quartic polynomial  $V(M)$ .

### 2. Critical behaviour for a nonsymmetric quartic polynomial

Let  $V_c(x)$  be a critical quartic polynomial,

$$V'_c(x) = \frac{1}{T_c}(x^3 - 4c_1x^2 + 2c_2x + 8c_1) \quad T_c = 1 + 4c_1^2 \quad V_c(0) = 0 \quad (2.1)$$

where we denote

$$c_k = \cos k\pi\epsilon \quad s_k = \sin k\pi\epsilon. \quad (2.2)$$

It corresponds to the critical density

$$\rho_c(x) = \frac{1}{2\pi T_c}(x - 2c_1)^2 \sqrt{4 - x^2}. \quad (2.3)$$

Observe that  $0 < \epsilon < 1$  is a parameter of the problem which determines the location of the critical point,

$$-2 < 2c_1 = 2 \cos \pi\epsilon < 2. \quad (2.4)$$

Equation (1.14) reads in this case as

$$\omega(z) = \frac{V'_c(z)}{2} - \frac{(z - 2c_1)^2 \sqrt{z^2 - 4}}{2T_c}. \quad (2.5)$$

The correlations between eigenvalues in the matrix model are expressed in terms of orthogonal polynomials  $P_n(x) = x^n + \dots$  on the line with respect to the weight  $e^{-NV_c(x)}$  (see, e.g., [Meh, TW]). Let

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} P_n(x) e^{-NV_c(x)/2} \quad n = 0, 1, \dots \quad (2.6)$$

be the corresponding psi-functions, which form an orthonormal basis in  $L^2$ . They satisfy the three-term recurrence relation,

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1} + \beta_n\psi_n + \gamma_n\psi_{n-1} \quad \gamma_n = \sqrt{\frac{h_n}{h_{n-1}}} \quad (2.7)$$

(see, e.g., [Sze]), and the differential equation,

$$\begin{aligned} \frac{1}{N}\psi'_n(x) + \frac{V'_c(x)}{2}\psi_n(x) &= \frac{n}{N}\frac{1}{\gamma_n}\psi_{n-1} + \frac{1}{T_c}\gamma_n\gamma_{n-1}(\beta_n + \beta_{n-1} + \beta_{n-2} - 4c_1)\psi_{n-2} \\ &+ \frac{1}{T_c}\gamma_n\gamma_{n-1}\gamma_{n-2}\psi_{n-3}. \end{aligned} \quad (2.8)$$

The differential equation can be written for any  $V$  (see [CI1, Eyn2] and references therein). The compatibility condition of equations (2.7), (2.8) leads to the string equations,

$$T_c \frac{n}{N} \frac{1}{\gamma_n^2} = \gamma_n^2 + \gamma_{n-1}^2 + \gamma_{n+1}^2 + \beta_n^2 + \beta_{n-1}^2 + \beta_n \beta_{n-1} - 4c_1(\beta_n + \beta_{n-1}) + 2c_2 \tag{2.9}$$

$$0 = V'_c(\beta_n) + \gamma_n^2(2\beta_n + \beta_{n-1}) + \gamma_{n+1}^2(2\beta_n + \beta_{n+1}) - 4c_1(\gamma_n^2 + \gamma_{n+1}^2). \tag{2.10}$$

To study the critical asymptotics we embed  $V_c(x)$  into a parametric family of polynomials. To that end for any  $T > 0$  we define the polynomial

$$V(x; T) = \frac{1}{T} V(x) \tag{2.11}$$

where  $V(x)$  is such that

$$V'(x) = x^3 - 4c_1x^2 + 2c_2x + 8c_1 \quad V(0) = 0. \tag{2.12}$$

Then  $V'_c(x) = V'(x; T_c)$ . We call  $T$  *temperature* and  $T_c$  *critical temperature*. Denote  $\Delta T = T - T_c$ . Let  $\rho(x; T)$  be the equilibrium density for the polynomial  $V(x; T)$ . Equation (1.14) reads in this case,

$$\omega(z; T) \equiv \int_{J(T)} \frac{\rho(x; T) dx}{z - x} = \frac{V'(z)}{2T} - \frac{h(z; T)R^{1/2}(z; T)}{2T} \tag{2.13}$$

where  $h(z; T)$  is a monic polynomial in  $z$ .

*Free energy near the critical point.* The (infinite volume) free energy is defined as

$$F(T) = -T \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln Z_N(T) \quad Z_N(T) = \int_{\mathcal{H}_N} \exp\left(-\frac{N}{T} \text{Tr } V(M)\right) dM. \tag{2.14}$$

We will show that at  $T = T_c$ ,  $F(T)$  is not analytic. To evaluate the type of nonanalyticity at  $T = T_c$ , consider the function

$$F_1(T) = T^2 \frac{d}{dT} \left( \frac{F(T)}{T} \right) = \lim_{N \rightarrow \infty} \frac{1}{Z_N(T)} \int_{\mathcal{H}_N} \frac{1}{N} \text{Tr } V(M) \exp\left(-\frac{N}{T} \text{Tr } V(M)\right) dM. \tag{2.15}$$

It can be evaluated as

$$\begin{aligned} F_1(T) &= \int_{J(T)} V(x)\rho(x; T) dx = -\frac{1}{4\pi iT} \oint_C V(z)h(z; T)\sqrt{R(z; T)} dz \\ &= \frac{1}{2\pi i} \oint_C V(z)\omega(z; T) dz \end{aligned} \tag{2.16}$$

where  $C$  is any contour with positive orientation around  $J(T)$ , the support of equilibrium measure. Observe that the limits in (2.14); (2.15) exist for any polynomial  $V(x)$  (cf [Joh]). In contrast, for the second derivative of  $F(T; N) \equiv -(T/N^2) \ln Z_N(T)$  in  $T$ , the convergence  $F''(T; N) \rightarrow F''(T)$  does *not* hold if the equilibrium measure has two cuts or more, because of quasiperiodic oscillations of  $F''(T; N)$  as a function of  $N$  (see [BDE]). From (2.16) it follows that since  $\omega(z; T)$  is continuous on  $C$  in  $T = T_c$ ,  $F_1(T)$  is continuous as well. Therefore,  $F'(T)$  is *continuous* at  $T = T_c$ . Consider  $F''(T)$ .

*Second derivative of the free energy.* From (2.16),

$$\frac{d[T F_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C V(z) \frac{d}{dT} [T \omega(z; T)] dz. \tag{2.17}$$

For  $T > T_c$ , the equilibrium measure corresponding to  $V(x; T)$  is supported by one cut  $[a, b]$  and the equilibrium density is written as

$$\rho(x; T) = \frac{1}{2\pi T} [(x - c)^2 + d^2] \sqrt{(b - x)(x - a)} \tag{2.18}$$

where  $a = a(T)$ ,  $b = b(T)$ ,  $c = c(T)$ ,  $d = d(T)$ . In the one-cut case we have the equation,

$$\frac{d}{dT}[T\omega(z; T)] = \frac{1}{\sqrt{(z-a)(z-b)}} \tag{2.19}$$

see [CI2] (or the appendix), hence

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{(z-a)(z-b)}} dz. \tag{2.20}$$

This implies that

$$\left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^+} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{z^2-4}} dz. \tag{2.21}$$

We should mention here another useful formula valid for  $T \geq T_c$  (see [DGZ, Eyn1, CI2]):

$$\frac{d^2[TF(T)]}{dT^2} = 2 \ln \frac{b-a}{4}. \tag{2.22}$$

For  $T < T_c$ , the equilibrium measure corresponding to  $V(x)$  is supported by two cuts  $[a_1, b_1]$  and  $[a_2, b_2]$ . The equilibrium density is written in this case as

$$\rho(x; T) = \frac{1}{2\pi T} (x-c) \sqrt{(b_1-x)(x-a_1)(b_2-x)(x-a_2)} \tag{2.23}$$

where  $a_1, b_1, a_2, b_2, c$  depend on  $T$  and  $b_1 < c < a_2$ . In the two-cut case we have the equation,

$$\frac{d}{dT}[T\omega(z; T)] = \frac{z-x_0}{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}} \tag{2.24}$$

where  $x_0 = x_0(T)$ ,  $b_1 < x_0 < a_2$ , is determined from the condition that

$$\int_{b_1}^{a_2} \frac{x-x_0}{\sqrt{(x-a_1)(x-b_1)(x-a_2)(x-b_2)}} dx = 0 \tag{2.25}$$

see the appendix, hence

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)(z-x_0)}{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}} dz. \tag{2.26}$$

We have that  $a_2 = b_1 = x_0 = 2c_1$  at  $T = T_c^-$ , hence

$$\left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^-} = \frac{1}{2\pi i} \oint_C \frac{V(z)}{\sqrt{z^2-4}} dz. \tag{2.27}$$

Thus,

$$\left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^-} = \left. \frac{d[TF_1(T)]}{dT} \right|_{T=T_c^+} \tag{2.28}$$

so that  $F''(T)$  is continuous at  $T = T_c$ . Consider now  $F'''(T)$ .

*Third derivative of the free energy.* In the one-cut case we have that

$$\frac{d}{dT}[(x-c)^2 + d^2] \sqrt{(b-x)(x-a)} = -\frac{2}{\sqrt{(b-x)(x-a)}} \tag{2.29}$$

and

$$\frac{da}{dT} = \frac{4}{h(a)(a-b)} \quad \frac{db}{dT} = \frac{4}{h(b)(b-a)} \quad h(x) = (x-c)^2 + d^2 \tag{2.30}$$

(see the appendix). From (1.18) we find that

$$c = 2c_1 - \frac{a+b}{4} \quad d^2 = \frac{5}{16}(a+b)^2 - c_1(a+b) - \frac{1}{2}ab - 2 \tag{2.31}$$



and then that

$$h(a)|_{a=-2,b=2} = 4(c_1 + 1)^2 \quad h(b)|_{a=-2,b=2} = 4(c_1 - 1)^2. \tag{2.32}$$

Therefore,  $a(T)$  and  $b(T)$  are analytic at  $T = T_c^+$ , as a solution of system (2.30) with analytic coefficients. Equation (2.20) implies that  $F_1(T)$ , and hence  $F(T)$ , are analytic at  $T = T_c^+$ . From (2.30) and (2.31) we obtain

$$\frac{da}{dT}\Big|_{T=T_c^+} = -\frac{1}{4(1+c_1)^2} \quad \frac{db}{dT}\Big|_{T=T_c^+} = \frac{1}{4(1-c_1)^2}. \tag{2.33}$$

The analyticity at  $T = T_c^-$  is more difficult. In the two-cut case we have that

$$\frac{d}{dT}(z-c)\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)} = -\frac{2(z-x_0)}{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}} \tag{2.34}$$

where  $b_1 < x_0 < a_2$  solves equation (2.25), and

$$\begin{aligned} \frac{da_1}{dT} &= \frac{4(a_1-x_0)}{(a_1-c)(a_1-b_1)(a_1-a_2)(a_1-b_2)} \\ \frac{db_2}{dT} &= \frac{4(b_2-x_0)}{(b_2-c)(b_2-a_1)(b_2-b_1)(b_2-a_2)} \end{aligned} \tag{2.35}$$

(see the appendix). At  $T = T_c^-$  this gives that

$$\frac{da_1}{dT}\Big|_{T=T_c^-} = -\frac{1}{4(1+c_1)^2} \quad \frac{db_2}{dT}\Big|_{T=T_c^-} = \frac{1}{4(1-c_1)^2}. \tag{2.36}$$

Define  $d$  and  $\delta$  such that

$$b_1 = c - d + \delta \quad a_2 = c + d + \delta. \tag{2.37}$$

Then  $d, \delta \rightarrow 0$  as  $T \rightarrow T_c^-$  and from (1.16),

$$\begin{aligned} 0 &= \int_{c-d+\delta}^{c+d+\delta} (x-c)\sqrt{(c+d+\delta-x)(x-c+d-\delta)}\sqrt{(x-a_1)(b_2-x)} \, dx \\ &= \int_{-d}^d (u+\delta)\sqrt{d^2-u^2}\sqrt{(u+c-a_1+\delta)(b_2-c-\delta-u)} \, du \\ &= d^3 \int_{-1}^1 \left(v + \frac{\delta}{d}\right) \sqrt{1-v^2}\sqrt{(vd+c-a_1+\delta)(b_2-c-\delta-vd)} \, dv \\ &= d^3 \sqrt{(c-a_1)(b_2-c)} \left[ \frac{d}{2} \frac{(a_1+b_2-2c)}{(c-a_1)(b_2-c)} \int_{-1}^1 v^2 \sqrt{1-v^2} \, dv \right. \\ &\quad \left. + \frac{\delta}{d} \int_{-1}^1 \sqrt{1-v^2} \, dv + O\left(d^3 + \frac{\delta^2}{d}\right) \right] \\ &= d^3 \sqrt{(c-a_1)(b_2-c)} \left[ \frac{\pi}{16} \frac{(a_1+b_2-2c)}{(c-a_1)(b_2-c)} d + \frac{\pi}{2} \frac{\delta}{d} + O\left(d^3 + \frac{\delta^2}{d}\right) \right] \end{aligned} \tag{2.38}$$

hence

$$\delta = -\frac{1}{8} \frac{(a_1+b_2-2c)}{(c-a_1)(b_2-c)} d^2 + O(d^4). \tag{2.39}$$

By the implicit function theorem,  $\delta$  is an analytic function of  $d^2$  at  $d = 0$ . Equation (1.18) gives that

$$\begin{aligned} c &= 2c_1 - \frac{a_1+b_2}{4} - \frac{1}{2}\delta \\ d^2 &= 2(1+c_1)da_1 - 2(1-c_1)db_2 - \frac{1}{2}\delta^2 - \frac{5}{8}(da_1^2 + db_2^2) - \frac{1}{4}da_1db_2 \\ da_1 &\equiv a_1 + 2 \quad db_2 \equiv -2 + b_2. \end{aligned} \tag{2.40}$$

Therefore, from (2.36) and (2.39) we obtain that as  $T \rightarrow T_c^-$ ,

$$\begin{aligned} d^2 &= -\frac{1}{s_1^2} \Delta T + O(\Delta T^2) & \delta &= -\frac{c_1}{8s_1^4} \Delta T + O(\Delta T^2) \\ c &= 2c_1 - \frac{3c_1}{16s_1^4} \Delta T + O(\Delta T^2) & \frac{b_1 + a_2}{2} &= 2c_1 - \frac{5c_1}{16s_1^4} \Delta T + O(\Delta T^2). \end{aligned} \tag{2.41}$$

Define now  $\delta_0$  such that

$$b_1 = x_0 - d + \delta_0 \quad a_2 = x_0 + d + \delta_0. \tag{2.42}$$

Then, similar to (2.38), we derive from (2.25) that

$$\begin{aligned} 0 &= \int_{x_0-d+\delta_0}^{x_0+d+\delta_0} \frac{x - x_0}{\sqrt{(x_0 + d + \delta_0 - x)(x - x_0 + d - \delta_0)} \sqrt{(x - a_1)(b_2 - x)}} dx \\ &= \frac{d}{\sqrt{(c - a_1)(b_2 - c)}} \left[ -\frac{\pi}{2} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} d + \pi \frac{\delta_0}{d} + O\left(d^3 + \frac{\delta_0^2}{d}\right) \right] \end{aligned} \tag{2.43}$$

hence

$$\delta_0 = \frac{1}{2} \frac{(a_1 + b_2 - 2c)}{(c - a_1)(b_2 - c)} d^2 + O(d^4) \tag{2.44}$$

so that as  $T \rightarrow T_c^-$ ,

$$\delta_0 = \frac{c_1}{2s_1^4} \Delta T + O(\Delta T^2). \tag{2.45}$$

By the implicit function theorem  $\delta_0$  is an analytic function of  $d^2$  at  $d = 0$ . Using (2.41) we obtain that

$$x_0 = 2c_1 - \frac{13c_1}{16s_1^4} \Delta T + O(\Delta T^2). \tag{2.46}$$

By (2.37) and (2.42),

$$x_0 = c + \delta - \delta_0, \tag{2.47}$$

hence equations (2.35) can be written as

$$\begin{aligned} \frac{da_1}{dT} &= \frac{4(a_1 - c - \delta + \delta_0)}{(a_1 - c)(d^2 - (a_1 - c - \delta)^2)(a_1 - b_2)} \\ \frac{db_2}{dT} &= \frac{4(b_2 - c - \delta + \delta_0)}{(b_2 - c)(d^2 - (b_2 - c - \delta)^2)(b_2 - a_2)}. \end{aligned} \tag{2.48}$$

Observe that  $d^2$  is an analytic function of  $a_1, b_2$ , and  $\delta, \delta_0$  are analytic functions of  $d^2$ . This gives the right-hand side in (2.48) as analytic functions of  $a_1, b_2$  and hence  $a_1, b_2$  are analytic as functions of  $T$  at  $T = T_c^-$ . Set

$$m = \frac{b_1 + a_2}{2}. \tag{2.49}$$

Then  $b_1 = m - d, a_2 = m + d$ , hence, by (2.26)

$$\frac{d[TF_1(T)]}{dT} = \frac{1}{2\pi i} \oint_C \frac{V(z)(z - x_0)}{\sqrt{(z - a_1)(z^2 - 2mz + m^2 - d^2)(z - b_2)}} dz. \tag{2.50}$$

Since  $x_0, m$  and  $d^2$  are analytic in  $T$  at  $T = T_c^-$ , we obtain that  $F(T)$  is analytic at  $T = T_c^-$ .

By (2.20),

$$\frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^+} = \frac{1}{2\pi i} \oint_C V(z) \frac{d}{dT} \left( \frac{1}{\sqrt{(z - a)(z - b)}} \right) \Big|_{T=T_c^+} dz \tag{2.51}$$

and by (2.50),

$$\begin{aligned} \frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^-} &= \frac{1}{2\pi i} \oint_C V(z) \left[ \frac{d}{dT} \left( \frac{1}{\sqrt{(z-a_1)(z-b_2)}} \right) \frac{(z-x_0)}{\sqrt{z^2-2mz+m^2-d^2}} \right. \\ &\quad \left. + \frac{1}{\sqrt{(z-a_1)(z-b_2)}} \frac{d}{dT} \left( \frac{(z-x_0)}{\sqrt{z^2-2mz+m^2-d^2}} \right) \right] \Big|_{T=T_c^-} dz. \end{aligned} \tag{2.52}$$

Observe that by (2.33), (2.36),

$$\frac{da}{dT} \Big|_{T=T_c^+} = \frac{da_1}{dT} \Big|_{T=T_c^-} = -\frac{1}{4(1+c_1)^2} \quad \frac{db}{dT} \Big|_{T=T_c^+} = \frac{db_2}{dT} \Big|_{T=T_c^-} = \frac{1}{4(1-c_1)^2} \tag{2.53}$$

hence

$$\begin{aligned} \frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^+} - \frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^-} &= -\frac{1}{2\pi i} \oint_C V(z) \frac{1}{\sqrt{z^2-4}} \frac{d}{dT} \left( \frac{(z-x_0)}{\sqrt{z^2-2mz+m^2-d^2}} \right) \Big|_{T=T_c^-} dz. \end{aligned} \tag{2.54}$$

From (2.41) and (2.46) we find that

$$\frac{d}{dT} \left( \frac{(z-x_0)}{\sqrt{z^2-2mz+m^2-d^2}} \right) \Big|_{T=T_c^-} = -\frac{1-c_1z+c_1^2}{s_1^4(z-2c_1)^2} \tag{2.55}$$

hence

$$\begin{aligned} \frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^+} - \frac{d^2[TF_1(T)]}{dT^2} \Big|_{T=T_c^-} &= \frac{1}{2\pi i} \oint_C V(z) \frac{1}{\sqrt{z^2-4}} \frac{1-c_1z+c_1^2}{s_1^4(z-2c_1)^2} dz \\ &= -\frac{3+25c_1^2+2c_1^4}{s_1^4} < 0. \end{aligned} \tag{2.56}$$

Thus,  $F(T)$  is analytic both at  $T = T_c^+$  and  $T = T_c^-$ , and  $F'''(T)$  has a jump at  $T = T_c$ . Therefore,  $T = T_c$  is a critical point of the *third-order* phase transition.

*Recurrence coefficients near the critical point.* The recurrence coefficients  $\gamma_n, \beta_n$  approach fixed values for  $T > T_c$  (see, e.g., [DKMVZ]). Namely,

$$\lim_{n, N \rightarrow \infty; \frac{n}{N} \rightarrow \frac{T}{T_c}} \gamma_n = \gamma(T) \quad \lim_{n, N \rightarrow \infty; \frac{n}{N} \rightarrow \frac{T}{T_c}} \beta_n = \beta(T) \tag{2.57}$$

where  $\gamma = \gamma(T), \beta = \beta(T)$  are fixed points of (2.9), (2.10), so that

$$T \frac{1}{\gamma^2} = 3\gamma^2 + 3\beta^2 - 8c_1\beta + 2c_2 \tag{2.58}$$

$$0 = V'(\beta) + 6\gamma^2\beta - 8c_1\gamma^2. \tag{2.59}$$

The values  $\gamma = \gamma(T), \beta = \beta(T)$  can be expressed in terms of the endpoints of the cut as

$$\gamma = \frac{b-a}{4} \quad \beta = \frac{b+a}{2} \tag{2.60}$$

see, e.g., [DGZ]. Therefore, by (2.33),

$$\gamma = 1 + \frac{1+c_1^2}{8s_1^4} \Delta T + O(\Delta T^2) \quad \beta = \frac{c_1}{2s_1^4} \Delta T + O(\Delta T^2) \quad \Delta T \rightarrow 0^+. \tag{2.61}$$

For  $T < T_c$ , the recurrence coefficients are asymptotically quasiperiodic, see [DKMVZ]. More precisely,

$$\lim_{n, N \rightarrow \infty; \frac{n}{N} \rightarrow \frac{T}{T_c}} [\gamma_n - \gamma(\omega n + \varphi)] = 0 \quad \lim_{n, N \rightarrow \infty; \frac{n}{N} \rightarrow \frac{T}{T_c}} [\beta_n - \beta(\omega n + \varphi)] = 0 \quad (T < T_c) \tag{2.62}$$

where

$$\omega = \omega(T) = 1 - \frac{1}{K} \int_{b_2}^{\infty} \frac{dz}{\sqrt{R(z)}} \quad K = \int_{b_1}^{a_2} \frac{dz}{\sqrt{R(z)}} \tag{2.63}$$

$$R(z) = (z - a_1)(z - b_1)(z - a_2)(z - b_2)$$

$\gamma(x) = \gamma(x; T)$ ,  $\beta(x) = \beta(x; T)$  are explicit analytic even periodic functions of period 1 in  $x$ , and  $\varphi$  is an explicit phase, see [BDE]. The extrema of  $\gamma(x)$  and  $\beta(x)$  are expressed in terms of the endpoints of the cuts,

$$\begin{aligned} \min_x \gamma(x) &= \frac{b_2 - a_1 - (a_2 - b_1)}{4} & \max_x \gamma(x) &= \frac{b_2 - a_1 + (a_2 - b_1)}{4} \\ \min_x \beta(x) &= \frac{b_2 + a_1 - (a_2 - b_1)}{2} & \max_x \beta(x) &= \frac{b_2 + a_1 + (a_2 - b_1)}{2}. \end{aligned} \tag{2.64}$$

Using (2.41), we obtain that as  $\Delta T \rightarrow 0^-$ ,

$$K = \frac{\pi}{2s_1} + O(\Delta T) \quad \int_{b_2}^{\infty} \frac{dz}{\sqrt{R(z)}} = \frac{\pi(1 - \varepsilon)}{2s_1} + O(\Delta T) \tag{2.65}$$

hence

$$\omega = \varepsilon + O(\Delta T) \quad \Delta T \rightarrow 0^-. \tag{2.66}$$

As regards the extrema of  $\gamma(x)$  and  $\beta(x)$ , they behave as

$$\min_x, \max_x \gamma(x) = 1 \pm \frac{1}{2s_1} \left( \frac{|\Delta T|}{2} \right)^{1/2} + \frac{1 + c_1^2}{8s_1^4} \Delta T + O(|\Delta T|^{3/2}) \tag{2.67}$$

$$\min_x, \max_x \beta(x) = \pm \frac{1}{s_1} \left( \frac{|\Delta T|}{2} \right)^{1/2} + \frac{c_1}{2s_1^4} \Delta T + O(|\Delta T|^{3/2}) \quad \Delta T \rightarrow 0^-. \tag{2.68}$$

2.1. Double scaling limit for recurrence coefficients

We considered above the case when we first took the limit  $n, N \rightarrow \infty, \frac{n}{N} \rightarrow \frac{T}{T_c}$ , and then the limit  $T \rightarrow T_c$ . Here we will consider the double scaling limit, when  $n, N \rightarrow \infty, \frac{n}{N} \rightarrow 1$ , with an appropriate scaling of  $n - N$ . We start with the following ansatz, which reproduces the quasiperiodic behaviour of the recurrence coefficients:

$$\frac{n}{N} = 1 + N^{-2/3} t \tag{2.69}$$

$$\begin{aligned} \gamma_n^2 &= 1 + N^{-1/3} u(t) \cos 2n\pi \varepsilon \\ &+ N^{-2/3} (v_0(t) + v_1(t) \cos 2n\pi \varepsilon + v_2(t) \cos 4n\pi \varepsilon) + N^{-1} (w_0(t) \\ &+ w_1(t) \cos 2n\pi \varepsilon + w_2(t) \cos 4n\pi \varepsilon + w_3(t) \cos 6n\pi \varepsilon + w_4(t) \sin 4n\pi \varepsilon) \end{aligned} \tag{2.70}$$

$$\begin{aligned} \beta_n &= 0 + N^{-1/3}u(t) \cos(2n + 1)\pi\epsilon \\ &\quad + N^{-2/3}(\tilde{v}_0(t) + \tilde{v}_1(t) \cos(2n + 1)\pi\epsilon + \tilde{v}_2(t) \cos(4n + 2)\pi\epsilon) \\ &\quad + N^{-1}(\tilde{w}_0(t) + \tilde{w}_1(t) \cos(2n + 1)\pi\epsilon + \tilde{w}_2(t) \cos(4n + 2)\pi\epsilon \\ &\quad + \tilde{w}_3(t) \cos(6n + 3)\pi\epsilon + \tilde{w}_4(t) \sin(4n + 2)\pi\epsilon) \end{aligned} \tag{2.71}$$

where  $u(t), v_0(t), \dots, \tilde{w}_4(t)$  are unknown functions. Observe that in the two-cut regime away from the critical point, the quasiperiodic behaviour of the recurrence coefficients is described by the Jacobi elliptic theta function ([DKMVZ]; see also [BDE]). In the double scaling limit, however, the theta function degenerates into a trigonometric function, and that is why asymptotics in (2.69) are expressed in trigonometric functions. We substitute ansatz (2.69) into string equations (2.9), (2.10) and equate terms of the same order.

*Order  $N^{-1/3}$ .* Our ansatz is automatically satisfied at this order.

*Order  $N^{-2/3}$ .* We obtain from (2.9), (2.10) that

$$\begin{pmatrix} c_1^2 + 1 & -2c_1 \\ -2c_1 & c_1^2 + 1 \end{pmatrix} \begin{pmatrix} v_0 \\ \tilde{v}_0 \end{pmatrix} = \frac{u^2}{4} \begin{pmatrix} c_1^2 \\ -c_1 \end{pmatrix} + \frac{T_c t}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.72}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \tilde{v}_1 \end{pmatrix} = \frac{u'}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.73}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1c_2 \\ -2c_1c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} v_2 \\ \tilde{v}_2 \end{pmatrix} = \frac{u^2}{4} \begin{pmatrix} 1 \\ -c_3 \end{pmatrix}. \tag{2.74}$$

By solving these equations we obtain that

$$v_0 = -\frac{c_1^2}{4s_1^2}u^2 + \frac{1 + c_1^2}{4s_1^4}tT_c \quad \tilde{v}_0 = -\frac{c_1}{4s_1^2}u^2 + tT_c \frac{c_1}{2s_1^4} \tag{2.75}$$

and

$$\tilde{v}_1 - v_1 = \frac{1}{2}u' \quad v_2 = \frac{u^2}{4s_1^2} \quad \tilde{v}_2 = \frac{c_1u^2}{4s_1^2}. \tag{2.76}$$

*Order  $N^{-1}$ .* We obtain from (2.9), (2.10) and (2.75), (2.76) that

$$\begin{pmatrix} c_1^2 + 1 & -2c_1 \\ -2c_1 & c_1^2 + 1 \end{pmatrix} \begin{pmatrix} w_0 \\ \tilde{w}_0 \end{pmatrix} = \frac{uu'}{4} \begin{pmatrix} 0 \\ -c_1 \end{pmatrix} + \frac{uv_1}{2} \begin{pmatrix} c_1^2 \\ -c_1 \end{pmatrix} + c_1 \begin{pmatrix} -\tilde{v}'_0 \\ v'_0 \end{pmatrix} \tag{2.77}$$

$$8c_1^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ \tilde{w}_1 \end{pmatrix} = \frac{u^3}{2s_1^2} \begin{pmatrix} -1 \\ c_2 \end{pmatrix} + \frac{T_c tu}{2s_1^4} \begin{pmatrix} 2c_1^4 + 3c_1^2 - 1 \\ -2c_1^4 - c_1^2 - 1 \end{pmatrix} + u'' \begin{pmatrix} -c_2 \\ 1 \end{pmatrix} + v'_1 \begin{pmatrix} -4c_1^2 \\ 4c_1^2 \end{pmatrix} \tag{2.78}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1c_2 \\ -2c_1c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} w_2 \\ \tilde{w}_2 \end{pmatrix} = \frac{uv_1}{2} \begin{pmatrix} 1 \\ -c_3 \end{pmatrix} + \frac{uu'}{4s_1^2} \begin{pmatrix} -2c_1^2c_2 \\ 4c_1^5 - 3c_1^3 + c_1 \end{pmatrix} \tag{2.79}$$

$$\begin{pmatrix} c_1^2 + c_3^2 & -2c_1c_3 \\ -2c_1c_3 & c_1^2 + c_3^2 \end{pmatrix} \begin{pmatrix} w_3 \\ \tilde{w}_3 \end{pmatrix} = \frac{u^3c_1^2}{8s_1^2} \begin{pmatrix} -c_2 + 2 \\ -2c_1c_3 + 1 \end{pmatrix} \tag{2.80}$$

$$\begin{pmatrix} c_1^2 + c_2^2 & -2c_1c_2 \\ -2c_1c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} w_4 \\ \tilde{w}_4 \end{pmatrix} = \frac{uu'}{8} \begin{pmatrix} -2s_2 \\ (4c_1^4 - 3c_1^2 - 1)/s_1 \end{pmatrix} \tag{2.81}$$

(we did symbolic calculations with MAPLE). Consider equation (2.78). The matrix on the left-hand side in this equation is degenerate, hence we have the compatibility condition,

$$\boxed{2s_1^2u'' = u^3 + \frac{T_c}{s_1^2}tu} \tag{2.82}$$

which is the Painlevé II equation. When  $\epsilon = 1/2$  it reduces to  $2u'' = u^3 + tu$ . The function  $u(t)$  behaves as

$$u(t) \underset{t \rightarrow -\infty}{\sim} \frac{1}{s_1} \sqrt{-T_c t} \quad u(t) \underset{t \rightarrow +\infty}{\sim} \text{Ai}(\kappa t) \quad \kappa = \left( \frac{T_c}{2s_1^4} \right)^{1/3}. \tag{2.83}$$

Here and in what follows we use the following notation:  $f(x) \sim g(x)$  as  $x \rightarrow a$  means that  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$ , and  $f(x) \approx g(x)$  as  $x \rightarrow a$  means that  $\lim_{x \rightarrow a} [f(x) - g(x)] = 0$ .

2.2. Scaled differential equations at the critical point

Equations (2.7), (2.8) can be used to derive a closed system of differential equations,

$$\frac{T_c}{N} \frac{d}{dx} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} -\frac{TV'(x)}{2} - \gamma_n^2 A_n(x) & \gamma_n B_n(x) \\ -\gamma_n B_{n-1}(x) & \frac{TV'(x)}{2} + \gamma_n^2 A_n(x) \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \tag{2.84}$$

where

$$A_n(x) = x - 4c_1 + \beta_n + \beta_{n-1} \tag{2.85}$$

$$B_n(x) = x^2 + x(\beta_n - 4c_1) + \beta_n^2 - 4c_1\beta_n + 2c_2 + \gamma_n^2 + \gamma_{n+1}^2. \tag{2.86}$$

To derive a scaled system at the critical point  $x = 2c_1$  we set

$$x = 2c_1 + yN^{-1/3}. \tag{2.87}$$

Then

$$\begin{aligned} \frac{T_c V'(x)}{2} + \gamma_n^2 A_n(x) &= 2c_1(1 - \gamma_n^2) + \gamma_n^2(\beta_n + \beta_{n-1}) \\ &+ (\gamma_n^2 - 1)yN^{-1/3} + c_1y^2N^{-2/3} + \frac{y^3N^{-1}}{2} \end{aligned} \tag{2.88}$$

$$B_n(x) = y^2 + \beta_n y + \gamma_n^2 + \gamma_{n+1}^2 - 2 - 2c_1\beta_n + \beta_n^2. \tag{2.89}$$

Substituting ansatz (2.69)–(2.71) we obtain that

$$\begin{aligned} \frac{T_c V'(x)}{2} + \gamma_n^2 A_n(x) &= N^{-2/3} \left( c_1y^2 + \frac{c_1u^2}{2} + \frac{c_1T_c t}{2s_1^2} + yu \cos 2n\pi\epsilon - s_1u' \sin 2n\pi\epsilon \right) \\ &+ O(N^{-1}) \\ \gamma_n B_n(x) &= N^{-2/3} \left( y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu \cos (2n + 1)\pi\epsilon + s_1u' \sin (2n + 1)\pi\epsilon \right) + O(N^{-1}) \\ \gamma_n B_{n-1} &= N^{-2/3} \left( y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu \cos (2n - 1)\pi\epsilon + s_1u' \sin (2n - 1)\pi\epsilon \right) + O(N^{-1}). \end{aligned} \tag{2.90}$$

Thus, system (2.84) reduces to

$$T_c \frac{d}{dy} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y) \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \tag{2.91}$$

where up to  $O(N^{-1/3})$ ,

$$a_{11}(y) = -c_1 \left( y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} \right) - yu \cos 2n\pi\epsilon + s_1u' \sin 2n\pi\epsilon \tag{2.92}$$

$$a_{12}(y) = y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2} + yu \cos(2n + 1)\pi\epsilon + s_1 u' \sin(2n + 1)\pi\epsilon \tag{2.93}$$

$$a_{21}(y) = -\left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}\right) - yu \cos(2n - 1)\pi\epsilon - s_1 u' \sin(2n - 1)\pi\epsilon \tag{2.94}$$

$$a_{22}(y) = c_1 \left(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}\right) + yu \cos 2n\pi\epsilon - s_1 u' \sin 2n\pi\epsilon. \tag{2.95}$$

When  $\epsilon = 1/2$ , this simplifies to

$$a_{11}(y) = -a_{22}(y) = -(-1)^n yu \tag{2.96}$$

$$a_{12}(y) = y^2 + \frac{u^2 + t}{2} + (-1)^n u' \tag{2.97}$$

$$a_{21}(y) = -\left(y^2 + \frac{u^2 + t}{2}\right) + (-1)^n u'. \tag{2.98}$$

Under the substitution

$$\psi_n(y) = \cos\left(n + \frac{1}{2}\right)\pi\epsilon f(y) - \sin\left(n + \frac{1}{2}\right)\pi\epsilon g(y) \tag{2.99}$$

$$\psi_{n-1}(y) = \cos\left(n - \frac{1}{2}\right)\pi\epsilon f(y) - \sin\left(n - \frac{1}{2}\right)\pi\epsilon g(y) \tag{2.100}$$

system (2.91) reduces, up to  $O(N^{-1/3})$ , to

$$\boxed{\frac{T_c}{s_1} \frac{d}{dy} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} s_1 u' & (y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}) + yu \\ -(y^2 + \frac{u^2}{2} + \frac{T_c t}{2s_1^2}) + yu & -s_1 u' \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}} \tag{2.101}$$

the differential  $\psi$ -equation for Painlevé II equation (2.82).

### 2.3. Universal kernel

To eliminate dependence on  $\epsilon$  consider new variables  $\tilde{t}$ ,  $\tilde{u}$  and  $\tilde{y}$  such that

$$t = \left(\frac{2s_1^4}{T_c}\right)^{1/3} \tilde{t} \quad u = \left(\frac{4T_c}{s_1}\right)^{1/3} \tilde{u} \quad y = \left(\frac{4T_c}{s_1}\right)^{1/3} \tilde{y}. \tag{2.102}$$

Then equations (2.82) and (2.101) reduce to

$$\tilde{u}'' = \tilde{t}\tilde{u} + 2\tilde{u}^3 \quad (') = \frac{d}{d\tilde{t}} \tag{2.103}$$

and

$$\frac{d}{d\tilde{y}} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 2\tilde{u}' & (4\tilde{y}^2 + 2\tilde{u}^2 + \tilde{t}) + 4\tilde{y}\tilde{u} \\ -(4\tilde{y}^2 + 2\tilde{u}^2 + \tilde{t}) + 4\tilde{y}\tilde{u} & -2\tilde{u}' \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.104}$$

Equations (2.102) give the scaling as

$$\frac{n}{N} = 1 + N^{-2/3} \left(\frac{2s_1^4}{T_c}\right)^{1/3} \tilde{t} \tag{2.105}$$

$$\gamma_n^2 = 1 + N^{-1/3} \left(\frac{4T_c}{s_1}\right)^{1/3} \tilde{u} \cos 2n\pi\epsilon + O(N^{-2/3}) \tag{2.106}$$

$$\beta_n = 0 + N^{-1/3} \left(\frac{4T_c}{s_1}\right)^{1/3} \tilde{u} \cos(2n + 1)\pi\epsilon + O(N^{-2/3}) \tag{2.107}$$

$$x = 2c_1 + N^{-1/3} \left(\frac{4T_c}{s_1}\right)^{1/3} \tilde{y}. \tag{2.108}$$

The Dyson integral kernel for the double scaling limit correlation functions is then

$$K(\tilde{y}_1, \tilde{y}_2) = \frac{f(\tilde{y}_1)g(\tilde{y}_2) - g(\tilde{y}_1)f(\tilde{y}_2)}{\tilde{y}_1 - \tilde{y}_2}. \tag{2.109}$$

### 3. Nonlinear hierarchy

#### 3.1. Basic ansatz

For  $m = 1, 2, \dots$ , we consider the model critical density

$$\rho(x) = \frac{1}{2\pi T_c} (x - 2c_1)^{2m} \sqrt{4 - x^2} \tag{3.1}$$

where

$$T_c = \frac{1}{2\pi} \int_{-2}^2 (x - 2c_1)^{2m} \sqrt{4 - x^2} dx. \tag{3.2}$$

The corresponding polynomial  $V(x)$  is such that

$$V'(x) = \frac{1}{T_c} \text{Pol}[(x - 2c_1)^{2m} \sqrt{4 - x^2}] \tag{3.3}$$

where  $\text{Pol}[f(x)]$  means a polynomial part of a function  $f(x)$  at infinity. In particular,

$$m = 1: \quad V'(x) = \frac{1}{T_c} [x^3 - 4c_1x^2 + 2c_2x + 8c_1] \quad T_c = 1 + 4c_1^2 \tag{3.4}$$

$$m = 2: \quad V'(x) = \frac{1}{T_c} [x^5 - 8c_1x^4 + (-2 + 24c_1^2)x^3 - 16c_1c_2x^2 + (-2 + 16c_1^4 - 48c_1^2)x + 16(c_1 + 4c_1^3)] \quad T_c = 2 + 24c_1^2 + 16c_1^4 \tag{3.5}$$

and so on. In fact, our considerations will be very general and (3.1) is only an example. They hold for any density (1.11) which satisfies regularity conditions (1.19), (1.20) everywhere except for one point  $c$  lying strictly inside one of the cuts, and such that as  $z \rightarrow c$ ,  $h(z) \sim C(x - c)^{2m}$ ,  $C \neq 0$ .

In the double scaling limit we define variables  $K$ ,  $t$  and  $y$  as

$$K = N^{-1/(2m+1)} \quad \frac{n}{N} = 1 + K^{2m} s_1 t \quad x = 2c_1 + 2Ky. \tag{3.6}$$

Our ansatz for the orthogonal polynomials is the following:

$$\begin{aligned} \psi(n, x) = & \cos(n + 1/2)\pi\epsilon f(t, y) - \sin(n + 1/2)\pi\epsilon g(t, y) \\ & + K [\cos(n + 1/2)\pi\epsilon f_1(t, y) - \sin(n + 1/2)\pi\epsilon g_1(t, y) \\ & + \cos 3(n + 1/2)\pi\epsilon \tilde{f}(t, y) - \sin 3(n + 1/2)\pi\epsilon \tilde{g}(t, y)] + O(K^2) \end{aligned} \tag{3.7}$$

[cf (2.99)], and for the recurrence coefficients,

$$\gamma_n = 1 + Ku(t) \cos 2n\pi\epsilon + O(K^2) \quad \beta_n = 2Ku(t) \cos (2n + 1)\pi\epsilon + O(K^2) \tag{3.8}$$

[cf (2.70), (2.71)]. See also [PeS] where an intimately related ansatz for the recurrence coefficients was suggested in the case of a *symmetric* potential  $V(x)$  in the circular ensemble. We substitute ansatz (3.7), (3.8) into the three-term recursion relation,

$$x\psi(n, x) = \gamma_{n+1}\psi(n + 1, x) + \beta_n\psi(n, x) + \gamma_n\psi(n - 1, x) \tag{3.9}$$



and in the first order in  $K$  we obtain two systems of equations,

$$\partial_t \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} = L \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} \quad L = \begin{pmatrix} 0 & y + u(t) \\ -y + u(t) & 0 \end{pmatrix} \quad (3.10)$$

(at frequency 1) and

$$\begin{pmatrix} \tilde{f}(t, y) \\ \tilde{g}(t, y) \end{pmatrix} = \frac{c_1 u(t)}{4s_1^2} \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} \quad (3.11)$$

(at frequency 3).

### 3.2. Differential system

We would like to derive a differential equation in  $y$ ,

$$\partial_y \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix} = D(t, y) \begin{pmatrix} f(t, y) \\ g(t, y) \end{pmatrix}. \quad (3.12)$$

We are looking for  $D(t, y)$  in the form

$$D(t, y) = \begin{pmatrix} -A(t, y) & yB(t, y) + C(t, y) \\ yB(t, y) - C(t, y) & A(t, y) \end{pmatrix} \quad (3.13)$$

[cf (2.101)], where  $A$ ,  $B$  and  $C$  are even polynomials in  $y$  of the following degrees:

$$\deg A = 2m - 2 \quad \deg B = 2m - 2 \quad \deg C = 2m. \quad (3.14)$$

We will assume that  $C$  is a monic polynomial, so that  $C = y^{2m} + \dots$ . The general case can be reduced to this one by the change of variables,  $t = \kappa \tilde{t}$ ,  $y = \frac{\tilde{y}}{\kappa}$ ,  $u(t) = \frac{\tilde{u}(\tilde{t})}{\kappa}$ , which preserves the structure of the operator  $L$  in (3.10). The consistency condition of equations (3.10) and (3.12),

$$[D, L] = \partial_y L - \partial_t D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \partial_t D \quad (3.15)$$

implies that

$$\partial_t B = 2A \quad \partial_t C = 1 + 2uA \quad \partial_t A = -2y^2 B + 2uC. \quad (3.16)$$

**Example.**  $m = 1$ . According to (3.14),  $A = a(t)$ ,  $B = b(t)$ ,  $C = y^2 + c(t)$ . From the last equation in (3.16) we obtain that  $b = u$  and then that

$$a = \frac{u'}{2} \quad b = u \quad c = t + \frac{u^2}{2} + t_0 \quad (3.17)$$

where  $t_0$  is a free constant, and

$$\frac{u''}{2} = u^3 + 2(t + t_0)u \quad (3.18)$$

the Painlevé II equation. By changing  $t + t_0$  to  $t$  we can reduce it to  $t_0 = 0$ .

We would like to construct solutions to (3.16) for  $m > 1$ . To that end, define recursively functions  $A_m(t, y)$ ,  $B_m(t, y)$ ,  $C_m(t, y)$  by the equations

$$C_{m+1} = y^2 C_m + f_m(u) \quad (3.19)$$

$$B_{m+1} = y^2 B_m + R_m(u) \quad (3.20)$$

$$A_{m+1} = y^2 A_m + \frac{1}{2} \partial_t R_m(u) \quad (3.21)$$

where  $R_m(u)$ ,  $f_m(u)$  solve the recursive equations

$$R_{m+1}(u) = u f_m(u) - \frac{1}{4} \partial_{tt} R_m(u) \tag{3.22}$$

$$\partial_t f_m(u) = u \partial_t R_m(u) \quad f_m(0) = 0 \tag{3.23}$$

with the initial data

$$A_0 = B_0 = 0 \quad C_0 = 1 \quad R_0(u) = u \quad f_0(u) = \frac{u^2}{2}. \tag{3.24}$$

We solve recursively (3.22)–(3.24) as

$$R_1(u) = \frac{1}{2}u^3 - \frac{1}{4}u'' \quad f_1(u) = \frac{3}{8}u^4 - \frac{1}{4}uu'' + \frac{1}{8}u'^2 \tag{3.25}$$

$$R_2(u) = \frac{3}{8}u^5 - \frac{5}{8}u^2u'' - \frac{5}{8}uu'^2 + \frac{1}{16}u^{(4)} \tag{3.26}$$

$$f_2(u) = \frac{5}{16}u^6 - \frac{5}{8}u^3u'' - \frac{5}{16}u^2u'^2 + \frac{1}{16}uu^{(4)} - \frac{1}{16}u'u''' + \frac{1}{32}u''^2 \tag{3.27}$$

$$R_3(u) = \frac{5}{16}u^7 - \frac{35}{32}u^4u'' - \frac{35}{16}u^3u'^2 + \frac{7}{32}u^2u^{(4)} + \frac{7}{8}uu'u''' + \frac{21}{32}uu''^2 + \frac{35}{32}u'^2u'' - \frac{1}{64}u^{(6)} \tag{3.28}$$

and so on,

$$R_m(u) = \frac{(2m)!}{2^{2m}(m!)^2}u^{2m+1} + \dots + \frac{(-1)^m}{2^{2m}}u^{(2m)}. \tag{3.29}$$

In addition,

$$A_1 = \frac{1}{2}u' \quad B_1 = u \quad C_1 = y^2 + \frac{1}{2}u^2 \tag{3.30}$$

$$A_2 = \frac{1}{2}u'y^2 + \frac{3}{4}u^2u' - \frac{1}{8}u''' \quad B_2 = uy^2 + \frac{1}{2}u^3 - \frac{1}{4}u'' \tag{3.31}$$

$$C_2 = y^4 + \frac{1}{2}u^2y^2 + \frac{3}{8}u^4 - \frac{1}{4}uu'' + \frac{1}{8}(u')^2 \tag{3.32}$$

and so on. It is easy to check that the functions  $A_m(t, y)$ ,  $B_m(t, y)$ ,  $C_m(t, y)$  defined by (3.19)–(3.24) solve the equations

$$\partial_t B_m = 2A_m \quad \partial_t C_m = 2uA_m \quad \partial_t A_m = -2y^2B_m + 2uC_m - 2R_m(u). \tag{3.33}$$

Indeed, by (3.24) it holds for  $m = 0$ . Assume that it holds for some  $m$ . Then by (3.19)–(3.23) and (3.33),

$$\partial_t C_{m+1} = y^2 \partial_t C_m + \partial_t f_m(u) = 2y^2uA_m + u \partial_t R_m(u) = 2uA_{m+1} \tag{3.34}$$

$$\partial_t B_{m+1} = y^2 \partial_t B_m + \partial_t R_m(u) = 2y^2A_m + \partial_t R_m(u) = 2A_{m+1} \tag{3.35}$$

$$\begin{aligned} \partial_t A_{m+1} &= y^2 \partial_t A_m + \frac{1}{2} \partial_{tt} R_m(u) = y^2[-2y^2B_m + 2uC_m - 2R_m(u)] + \frac{1}{2} \partial_{tt} R_m(u) \\ &= -2y^2[y^2B_m + R_m(u)] + 2u[y^2C_m + f_m(u)] - 2\left(u f_m(u) - \frac{1}{4} \partial_{tt} R_m(u)\right) \\ &= -2y^2B_{m+1} + 2uC_{m+1} - 2R_{m+1}(u) \end{aligned} \tag{3.36}$$

which proves (3.33) for  $m + 1$  and hence for all  $m = 0, 1, 2, \dots$ . Comparing (3.16) with (3.33) we obtain that

$$A = A_m \quad B = B_m \quad C = t + C_m \tag{3.37}$$

solve equation (3.16), provided  $u(t)$  is a solution of the equation

$$R_m(u) + tu = 0. \tag{3.38}$$

The sequence of equations (3.38) for  $m = 1, 2, \dots$  forms a hierarchy of ordinary differential equations which is known as the Painlevé II hierarchy [Kit] (see also [Moo, PeS]). We can now formulate the following result.

**Theorem 3.1.** *Define*

$$D_m(t, y) = \begin{pmatrix} -A_m(t, y) & yB_m(t, y) + C_m(t, y) \\ yB_m(t, y) - C_m(t, y) & +A_m(t, y) \end{pmatrix}. \quad (3.39)$$

Then if  $u(t)$  is a solution of equation (3.38), then the matrix

$$D(t, y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + D_m(t, y) \quad (3.40)$$

is a solution to (3.15). More generally, if  $t_1, \dots, t_m$  are arbitrary constants and  $u(t)$  is a solution of the equation

$$\sum_{k=1}^m t_k R_k(u) + tu = 0 \quad (3.41)$$

then the matrix

$$D(t, y) = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \sum_{k=1}^m t_k D_k(t, y) \quad (3.42)$$

is a solution to (3.15).

**Remark.** It can be shown that (3.42) is a *general* solution to equation (3.15).

The meaning of the constants  $t_1, \dots, t_m$  in (3.42) is the following. Observe that the differential equation in  $y$ , (3.12) describes the double scaling limit for a critical polynomial of degeneracy  $2m$ . In this case the space of transversal fluctuations to the manifold of critical polynomials has dimension  $m$ . The variables  $t_1, \dots, t_m$  serve as coordinates in the space of transversal fluctuations, and (3.42) gives the matrix describing the double scaling limit of the recurrence coefficients in the direction  $\tau = (t_1, \dots, t_m)$ .

#### 4. Conclusion

In this paper, we considered critical polynomials which violate the regularity conditions at exactly one point, inside the support of the equilibrium measure. It is characterized by the degree  $2m$  of degeneracy of the equilibrium density at the critical point. Our main results are the following:

- When  $m = 1$ , the infinite volume free energy exhibits the phase transition of the third order. This extends the result of [GW] to nonsymmetric critical polynomials.
- When  $m = 1$ , the double scaling limit of the recurrence coefficients is described, under a proper substitution, by the Hastings–McLeod solution to the Painlevé II differential equation. Earlier this result was known only for symmetric critical polynomials [DSS, PeS] (for rigorous results see [BI2, BDJ]).
- For  $m > 1$ , we derive a hierarchy of ordinary differential equations describing the double scaling limit of the recurrence coefficients.

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**Appendix. One useful identity**

Let  $V(z; T) = \frac{V(z)}{T}$ , where  $T > 0$  is the temperature, and

$$\begin{aligned} \mu_N(dM; T) &= Z_N(T)^{-1} \exp\left(-\frac{N}{T} \text{Tr } V(M)\right) dM \\ Z_N(T) &= \int_{\mathcal{H}_N} \exp\left(-\frac{N}{T} \text{Tr } V(M)\right) dM. \end{aligned} \tag{A.1}$$

Then  $\rho(x)$  and  $\omega(z)$  depend on  $T$ . The following identity is useful in many questions.

**Proposition.** *Assume that the number of cuts does not change in a neighbourhood of a given  $T > 0$ . Then*

$$\frac{d}{dT}[T\omega(z)] = \frac{\prod_{j=1}^{q-1}(z - x_j)}{R^{1/2}(z)} \tag{A.2}$$

where the numbers  $b_j < x_j < a_{j+1}$ ,  $j = 1, \dots, q - 1$ , solve the equations,

$$\int_{b_k}^{a_{k+1}} \frac{\prod_{j=1}^{q-1}(x - x_j)}{R^{1/2}(x)} dx = 0 \quad k = 1, \dots, q - 1. \tag{A.3}$$

The neighbourhood can be one-sided, then the derivative in  $T$  is also one-sided.

**Remark.** See also the recent paper [CG], where some general formulae are derived for the variation of the eigenvalue density function under the variation of  $V$ .

**Proof.** Equation (2.13) gives that

$$T\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}. \tag{A.4}$$

Since  $V(z)$  does not depend on  $T$ ,

$$\frac{d}{dT}[T\omega(z)] = -\frac{d}{dT} \frac{h(z)R^{1/2}(z)}{2}. \tag{A.5}$$

The function on the right-hand side can be written as

$$-\frac{d}{dT} \frac{Th(z)R^{1/2}(z)}{2} = \frac{P(z)}{R^{1/2}(z)} \tag{A.6}$$

where  $P(z)$  is a polynomial with real coefficients. Since

$$\frac{d}{dT}[T\omega(z)] = \frac{1}{z} + O(z^{-2}) \tag{A.7}$$

we obtain that

$$\frac{P(z)}{R^{1/2}(z)} = \frac{1}{z} + O(z^{-2}) \tag{A.8}$$

which shows that  $P(z) = z^{q-1} + \dots$ . By (1.16),

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2} dx = 0 \quad j = 1, \dots, q - 1. \tag{A.9}$$

By differentiating with respect to  $T$  we obtain that

$$\int_{b_j}^{a_{j+1}} \frac{P(x)}{R^{1/2}(x)} dx = 0 \quad j = 1, \dots, q-1. \quad (\text{A.10})$$

This is possible only if  $P(x)$  has a zero in each interval  $[b_j, a_{j+1}]$ . Thus, (A.3) is proved.  $\square$

As a corollary, from (A.5) we get that

$$\frac{d}{dT} [h(z)R^{1/2}(z)] = -\frac{2 \prod_{j=1}^{q-1} (z - x_j)}{R^{1/2}(z)}. \quad (\text{A.11})$$

This implies that

$$\frac{d}{dT} \ln[h(z)R^{1/2}(z)] = -\frac{2 \prod_{j=1}^{q-1} (z - x_j)}{h(z)R(z)}. \quad (\text{A.12})$$

Comparing the residue of both sides at  $z = a_k, b_k$  we obtain that

$$\begin{aligned} \frac{da_k}{dT} &= \frac{4 \prod_{j=1}^{q-1} (a_k - x_j)}{h(a_k)(a_k - b_k) \prod_{j:j \neq k} [(a_k - a_j)(a_k - b_j)]} \\ \frac{db_k}{dT} &= \frac{4 \prod_{j=1}^{q-1} (a_k - x_j)}{h(b_k)(b_k - a_k) \prod_{j:j \neq k} [(b_k - a_j)(b_k - b_j)]} \quad 1 \leq k \leq q. \end{aligned} \quad (\text{A.13})$$

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