Journal of Statistical Physics, Vol. 106, Nos. 1/2, January 2002 (© 2002)

$SU(1, 1)$ Random Polynomials

Pavel Bleher[1] **and Denis Ridzal**[1]

Received April 3, 2001; accepted July 5, 2001

We study statistical properties of zeros of random polynomials and random analytic functions associated with the pseudoeuclidean group of symmetries $SU(1, 1)$, by utilizing both analytical and numerical techniques. We first show that zeros of the $SU(1, 1)$ random polynomial of degree N are concentrated in a narrow annulus of the order of N^{-1} around the unit circle on the complex plane, and we find an explicit formula for the scaled density of the zeros distribution along the radius in the limit $N \to \infty$. Our results are supported through various numerical simulations. We then extend results of Hannay[(1)] and Bleher *et al.*[(2)] to derive different formulae for correlations between zeros of the $SU(1, 1)$ random analytic functions, by applying the generalized Kac–Rice formula. We express the correlation functions in terms of some Gaussian integrals, which can be evaluated combinatorially as a finite sum over Feynman diagrams or as a supersymmetric integral. Due to the $SU(1, 1)$ symmetry, the correlation functions depend only on the hyperbolic distances between the points on the unit disk, and we obtain an explicit formula for the two point correlation function. It displays quadratic repulsion at small distances and fast decay of correlations at infinity. In an appendix to the paper we evaluate correlations between the outer zeros $|z_j| > 1$ of the $SU(1, 1)$ random polynomial, and we prove that the inner and outer zeros are independent in the limit when the degree of the polynomial goes to infinity.

KEY WORDS: Random polynomial; pseudo-sphere; correlations between zeros.

1. INTRODUCTION

In this paper we are interested in statistical properties of zeros of random polynomials and random analytic functions associated with the pseudo-euclidean group of symmetries $SU(1, 1)$. The motivation for the study of zeros of random polynomials and random analytic functions comes from

[1] Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis, 402 N. Blackford Street, Indianapolis, Indiana 46202-3216; e-mail: pbleher@math.iupui.edu, dridzal@iupui.edu

different applications, most importantly from the theory of quantum chaos (see papers of Bogomolny *et al.*,[3] Leboeuf and Shukla,[4] Hannay,[1] Korsch *et al.*,[5] Nonnenmacher and Voros,[6] Forrester and Honner,[7] Leboeuf,[8] Shiffman and Zelditch,[9] and others). There are different ensembles of random polynomials associated with different groups of symmetries, like in the theory of random matrices (see ref. 10), although the symmetry plays somewhat different role in the ensembles of random matrices and random polynomials. The $O(n+1)$ ensemble of random polynomials consists of multivariate homogeneous real polynomials of $(n+1)$ variable $z=(z_0, z_1,..., z_n)$ of the form

$$\psi(z)=\sum_{|m|=N}\sqrt{C_N^m}\,a_m z^m, \tag{1}$$

where $m=(m_0, m_1,..., m_n)$ is a multiindex, $|m|=m_0+m_1+\cdots+m_n$, $z^m=z_0^{m_0}z_1^{m_1}\cdots z_n^{m_n}$, C_N^m is the multinomial coefficient,

$$C_N^m\equiv\frac{N!}{m_0!\,m_1!\cdots m_n!}, \tag{2}$$

and a_m are independent standard, $N(0, 1)$, real Gaussian random variables. Consider the set Z of joint real zeros of k, $k\leqslant n$, independent copies of the $O(n+1)$ random polynomial in the projective space $\mathbb{RP}^n$. This is a random real algebraic variety of dimension $n-k$. It is nondegenerate almost surely and it possesses a natural volume element induced by the standard metric in $\mathbb{RP}^n$. The joint distribution functions of the zeros are invariant with respect to the action of the group $O(n+1)$, and, in particular, the density function of the zeros is constant (cf. refs. 2, 11, and 12).

Similarly, the $SU(n+1)$ ensemble of random polynomials consists of multivariate homogeneous complex polynomials of $(n+1)$ complex variable $z=(z_0, z_1,..., z_n)$ of form (1) where a_m are independent standard complex Gaussian random variables (cf. refs. 1–3, 12–14). It corresponds to the unitary ensemble of random matrices. The distribution of joint zeros of k independent copies of the $SU(n+1)$ random polynomial is invariant with respect to the action of the group $SU(n+1)$ on the complex projective space $\mathbb{CP}^n$. As the degree N of the $SU(n+1)$ random polynomial goes to infinity, the scaled correlation functions of zeros approach a limit, which is represented by the correlation functions of the W_n ensemble of random analytic functions (see refs. 1, 2, 8, 12–14). The latter consists of random functions of the form

$$\psi(z)=\sum_m\sqrt{\frac{1}{m!}}\,a_m z^m, \tag{3}$$

where the sum runs over multiindices $m = (m_1,..., m_n)$ with $m_j \geqslant 0$, $m! = m_1!,..., m_n!$, and a_m are independent standard complex Gaussian random variables. The existence of the limit for the scaled correlation functions of zeros is valid in a very general framework of random sections of powers of line bundles over compact manifolds and the limit is universal (see refs. 2, 12–14). The explicit combinatorial formulae for the limit of two-point correlation functions are obtained in refs. 2, 12–14. They demonstrate a quadratic repulsion if $k = n = 1$, neutrality if $k = n = 2$, and attraction if $k = n > 2$.

In this paper we are interested in the pseudoeuclidean, $SU(1, 1)$ ensemble of random analytic functions. The general $SU(n, 1)$ ensemble consists of multivariate analytic functions of the form

$$\psi(z) = \sum_m \sqrt{C^m_{|m|+L-1}}\, a_m z^m, \qquad z = (z_1,..., z_n), \tag{4}$$

where $L \geqslant 1$ is a fixed integer, a parameter of the ensemble, the sum runs over multiindices $m = (m_1,..., m_n)$ with $m_j \geqslant 0$, $z^m = z_1^{m_1} \cdots z_n^{m_n}$,

$$C^m_{|m|+L-1} \equiv \frac{(|m|+L-1)!}{(L-1)!\, m_1! \cdots m_n!}, \tag{5}$$

and a_m are independent standard complex Gaussian random variables. We will restrict our study to the case $n = 1$. The basic calculations are extended to the case of any n, and we are going to return to this extension in subsequent publications. Another possible extension concerns the universality result like in refs. 2, 12, and 15. We would like also to mention here the recent works by Jancovici and Téllez,[16] who consider the one-component plasma on the pseudosphere, and by Zyczkowski and Sommers,[17] in which truncated random unitary matrices are shown to have an eigenvalue density uniform on the pseudosphere.

The plan of the paper is the following. In Section 2 we consider the ensemble of $SU(1, 1)$ random polynomials, which is obtained by restricting m in (4) to be bounded by N (with $n = 1$). We calculate the density function for the distribution of zeros of the $SU(1, 1)$ random polynomial and we show that the zeros are concentrated in a narrow annulus around the unit circle, of the width of the order of $1/N$. We find the scaled profile of the density function along the radius in the limit $N \to \infty$. In Section 2.2 we rederive the result of Leboeuf[8] for the density of the $SU(1, 1)$ random analytic function. In Section 3 we derive different formulae for the correlation functions between zeros of the $SU(1, 1)$ random analytic function. First, we apply the general result of ref. 2 to get the Kac–Rice type expression for the n-point correlation function. We then specify it for the case of

Gaussian random coefficients (cf. refs. 1, 2, and 18). This expresses the n-point correlation function in terms of some Gaussian averages, which is further represented as a sum over Feynman diagrams, or in a different approach developed in ref. 14, as a supersymmetric integral. In Section 3.3 we carry out concrete calculations for the two point correlation function in the $SU(1,1)$ ensemble. They demonstrate a quadratic repulsion at small distances and fast decay at infinity. In the limit when the main parameter L of the $SU(1,1)$ polynomial goes to infinity we recover the W_1 correlation function of Hannay. There is an appendix at the end of the paper, in which we consider correlations between outer zeros z_j, $|z_j|>1$, of the $SU(1,1)$ random polynomial. We show that in the limit $N\to\infty$, the outer zeros are independent of the inner zeros $|z_j|<1$, and after changing the variable z to $1/z$, the correlations between outer zeros coincide with the correlations between zeros in the $SU(1,1)$ ensemble with $L=1$.

2. BASIC STATISTICAL PROPERTIES

2.1. Scaled Density Function

Consider the following random polynomial, associated with the pseudo-euclidean group $SU(1,1)$:

$$\psi(z)=\sum_{m=0}^{N}\sqrt{C_{m+L-1}^{m}}\,a_m z^m,\qquad C_{m+L-1}^{m}\equiv\frac{(m+L-1)!}{(L-1)!\,m!},\tag{6}$$

where L is a fixed positive integer, a parameter of the problem, C_{m+L-1}^{m} are Newton's binomial coefficients, and a_m are independent standard complex Gaussian random variables, so that

$$Ea_m=0,\qquad Ea_m\overline{a_n}=\delta_{mn},\qquad Ea_m a_n=0.\tag{7}$$

When $L=1$, (6) reduces to the classical form

$$\psi(z)=\sum_{m=0}^{N}a_m z^m.\tag{8}$$

We wish to investigate the density of zeros $p_N(z)$ for the $SU(1,1)$ polynomial (6) as $N\to\infty$. The density $p_N(z)$ is determined by the condition that for any test function $\varphi(z)$, which is infinitely differentiable and compactly supported,

$$E\left(\sum_{j=1}^{N}\varphi(z_j)\right)=\int_{\mathbb{C}}p_N(z)\,\varphi(z)\,dz,\qquad dz\equiv dx\,dy,\tag{9}$$

where z_j are zeros of the random polynomial $\psi(z)$. Since the total number of zeros is equal to N, we have that

$$\int_{\mathbb{C}} p_N(z)\, dz = N. \tag{10}$$

Observe that the polynomial

$$\psi(e^{i\theta} z) = \sum_{m=0}^{N} \sqrt{C_{m+L-1}^{m}}\, a_m e^{im\theta} z^m \tag{11}$$

has the same probability distribution as $\psi(z)$, hence

$$p_N(re^{i\theta}) = p_N(r). \tag{12}$$

The general formula for $p_N(z)$ is given by the Poincaré–Lelong type expression

$$p_N(z) = \frac{1}{\pi} \frac{\partial^2}{\partial z\, \partial \bar{z}} [\ln E(\psi(z)\, \overline{\psi(z)})] \tag{13}$$

(see refs. 8 and 13). From (6),

$$\begin{aligned} E(\psi(z)\, \overline{\psi(z')}) &= \sum_{m=0}^{N} C_{m+L-1}^{m} (z\overline{z'})^m \\ &= \frac{1}{(L-1)!} \sum_{m=0}^{N} (m+1) \cdots (m+L-1)(z\overline{z'})^m. \end{aligned} \tag{14}$$

In particular,

$$\begin{aligned} E(\psi(z)\, \overline{\psi(z)}) = \mathscr{F}_{N,L}(x) &\equiv \sum_{m=0}^{N} C_{m+L-1}^{m} x^m \\ &= \frac{1}{(L-1)!} \sum_{m=0}^{N} (m+1) \cdots (m+L-1)\, x^m. \end{aligned} \tag{15}$$

where here and in what follows we use the notation

$$x = z\bar{z} = |z|^2. \tag{16}$$

It is obvious that

$$\mathcal{F}_{N,L}(x) > 0, \qquad \forall x \geqslant 0. \tag{17}$$

Substituting (15) into (13) gives that

$$p_N(z) = \frac{1}{\pi}\left[x\left(\frac{\mathcal{F}'_{N,L}(x)}{\mathcal{F}_{N,L}(x)}\right)' + \frac{\mathcal{F}'_{N,L}(x)}{\mathcal{F}_{N,L}(x)}\right]. \tag{18}$$

We further define

$$\mathcal{G}_{N,L}(x) = \sum_{m=0}^{N} x^{m+L-1} = x^{L-1}\frac{x^{N+1}-1}{x-1}. \tag{19}$$

Then from (15),

$$\mathcal{F}_{N,L}(x) = \frac{1}{(L-1)!}\frac{d^{L-1}\mathcal{G}_{N,L}(x)}{dx^{L-1}}. \tag{20}$$

Numerical simulations show that most of the roots of $SU(1,1)$ polynomials (6) are concentrated in a small annulus (of the width of the order of $1/N$) near the unit circle. This property is illustrated in Fig. 1, which contains a point-plot of the zeros of 150 $SU(1,1)$ polynomials of degree $N=200$, with $L=30$.

To get the asymptotics of the density $p_N(z)$ near the unit circle, we introduce a scaling of the variable x in the form

$$x = 1 + \frac{s}{N}. \tag{21}$$

Then,

$$\frac{d}{dx} = N\frac{d}{ds}. \tag{22}$$

In subsequent calculations, we will assume that

$$-A \leqslant s \leqslant A, \tag{23}$$

for some arbitrary fixed $A > 0$. The notation $R(s) = O(N^{-1})$ used below means that there exists some $C(A) > 0$ so that $|R(s)| \leqslant C(A)\, N^{-1}$ for all $s \in [-A, A]$. The main result of this section is the following theorem.

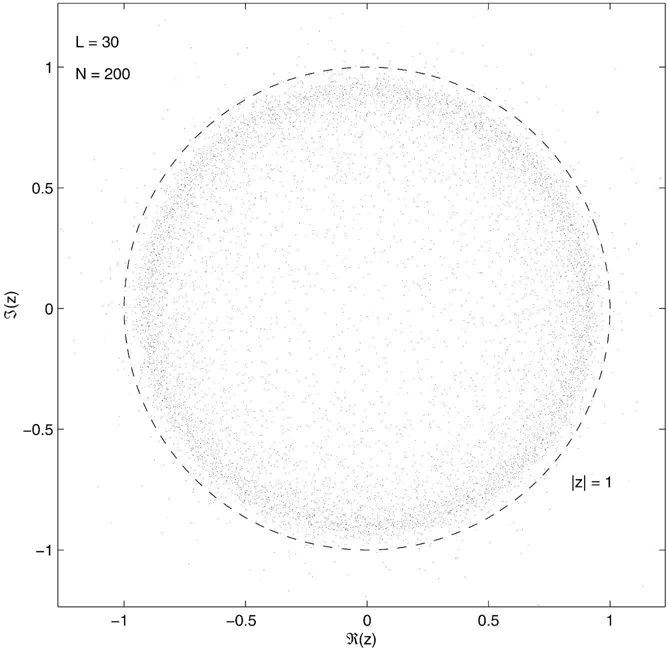

Fig. 1. Zeros of $SU(1, 1)$ random polynomials.

Theorem 2.1. As $N \to \infty$,

$$N^{-2} p_N\left(\left(1+\frac{s}{N}\right)^{1/2} e^{i\theta}\right) = \frac{1}{\pi}\left[\frac{g^{(L)}(s)}{g^{(L-1)}(s)}\right]' + O(N^{-1}). \tag{24}$$

where

$$g^{(j)}(s) = \frac{d^j}{ds^j}\left(\frac{e^s - 1}{s}\right). \tag{25}$$

Remark. Observe that one N in the normalization of $p_N(z)$ on the left is due to the rescaling (21) and one is due to the integral condition (10).

Proof. By substituting (21) into (19) we obtain that as $N \to \infty$,

$$\begin{aligned}\mathscr{G}_{N,L}(x) &= \left(1+\frac{s}{N}\right)^{L-1} \frac{(1+\frac{s}{N})^{N+1} - 1}{\frac{s}{N}} = N\,\frac{e^s - 1}{s}\,(1 + O(N^{-1})) \\ &= N g(s)(1 + O(N^{-1})),\end{aligned} \tag{26}$$

where we define

$$g(s) = \frac{e^s - 1}{s} = \sum_{j=0}^{\infty} \frac{s^j}{(j+1)!}. \tag{27}$$

Observe that for $j = 0, 1, 2, \ldots$,

$$g^{(j)}(s) \equiv \frac{d^j g(s)}{ds^j} > 0, \qquad -\infty < s < \infty. \tag{28}$$

It is obvious from (27) for $s \geqslant 0$. For negative s, use the identity

$$g^{(j)}(-s) = \frac{j!}{s^{j+1}} \left[1 - e^{-s} \left(1 + \frac{s}{1!} + \cdots + \frac{s^j}{j!} \right) \right]. \tag{29}$$

Formula (26) holds obviously in a small complex neighborhood of the segment $-A \leqslant s \leqslant A$. Therefore, we can differentiate it in s, so that

$$\frac{d^j \mathcal{G}_{N,L}(x)}{dx^j} = N^{j+1} g^{(j)}(s)(1 + O(N^{-1})), \tag{30}$$

Thus, from (20), we obtain that

$$\mathcal{F}_{N,L}(x) = \frac{N^L g^{(L-1)}(s)}{(L-1)!} (1 + O(N^{-1})), \qquad g^{(L-1)}(s) \equiv \frac{d^{L-1} g(s)}{ds^{L-1}}, \tag{31}$$

and

$$\mathcal{F}'_{N,L}(x) = \frac{N^{L+1} g^{(L)}(s)}{(L-1)!} (1 + O(N^{-1})), \tag{32}$$

hence

$$\frac{\mathcal{F}'_{N,L}(x)}{\mathcal{F}_{N,L}(x)} = \frac{N g^{(L)}(s)}{g^{(L-1)}(s)} (1 + O(N^{-1})). \tag{33}$$

Let us go back to formula (18). In this formula, $x = 1 + O(N^{-1})$, and the first term in the brackets has an extra derivative so this is the leading term, and we can neglect the second term. This gives that

$$N^{-2} p_N(z) = \frac{1}{\pi} \left[\frac{g^{(L)}(s)}{g^{(L-1)}(s)} \right]' + O(N^{-1}), \qquad z\bar{z} = 1 + \frac{s}{N}, \tag{34}$$

which was stated. Theorem 2.1 is proved.

Equation (24) can also be expressed in the following form:

$$N^{-2}p_N(z) = \frac{1}{\pi}[\log g^{(L-1)}(s)]'' + O(N^{-1}). \tag{35}$$

From (24),

$$\lim_{N\to\infty} N^{-2}p_N\left(\left(1+\frac{s}{2N}\right)e^{i\theta}\right) = p(s) \equiv \frac{1}{\pi}\left[\frac{g^{(L)}(s)}{g^{(L-1)}(s)}\right]'$$
$$= \frac{1}{\pi}[\log g^{(L-1)}(s)]''. \tag{36}$$

Consider now the distribution function of zeros. Define

$$P_N(x) = N^{-1}E\#\{j: |z_j|^2 \leqslant x\}, \tag{37}$$

which gives the expected value of the fraction of zeros in the disk of radius $\sqrt{x}$. Then

$$P_N'(x) = N^{-1}\pi p_N(z), \qquad x = |z|^2. \tag{38}$$

Hence (24) implies that

$$P_N\left(1+\frac{s}{N}\right) = \frac{g^{(L)}(s)}{g^{(L-1)}(s)} + O(N^{-1}). \tag{39}$$

As follows from (29), the limiting distribution function,

$$\lim_{N\to\infty} P_N\left(1+\frac{s}{N}\right) = P(s) \equiv \frac{g^{(L)}(s)}{g^{(L-1)}(s)}, \tag{40}$$

has the following asymptotics at $-\infty$:

$$P(s) = -\frac{L}{s} + O((-s)^{L-1}e^s), \qquad s \to -\infty. \tag{41}$$

At $+\infty$ we use the formula

$$g^{(j)}(s) = \frac{e^s}{s}\left[1 - \frac{j}{s} + \frac{j(j-1)}{s^2} - \cdots + \frac{(-1)^j j!}{s^j}\right] + \frac{(-1)^{j+1} j!}{s^{j+1}}, \tag{42}$$

which gives that

$$1-P(s)=\frac{1}{s}-\frac{L-1}{s^2}+O(s^{-3}), \qquad s\to\infty. \tag{43}$$

For $L=1$, (40) reduces to the well-known result (see ref. 19)

$$P(s)=\frac{se^s-e^s+1}{s(e^s-1)}. \tag{44}$$

For $L=1$, the limiting distribution is symmetric, so that

$$P(1-s)=1-P(s) \tag{45}$$

(it is related to the symmetry of the zeros of $\psi(z)$ in (8) with respect to the transformation $z\to 1/z$). Since

$$g(s)=1+\frac{s}{2!}+\frac{s^2}{3!}+\frac{s^3}{4!}+\cdots, \tag{46}$$

we have that

$$g^{(j)}(s)=\frac{1}{j+1}+\frac{s}{1!\,(j+2)}+\frac{s^2}{2!\,(j+3)}+\frac{s^3}{3!\,(j+4)}+\cdots. \tag{47}$$

Thus,

$$P_N\left(1+\frac{s}{N}\right)=\frac{\frac{1}{L+1}+\frac{s}{1!\,(L+2)}+\frac{s^2}{2!\,(L+3)}+\cdots}{\frac{1}{L}+\frac{s}{1!\,(L+1)}+\frac{s^2}{2!\,(L+2)}+\cdots}+O(N^{-1}). \tag{48}$$

In particular,

$$P_N(1)=\frac{L}{L+1}+O(N^{-1}), \tag{49}$$

which means that the expected value of the fraction of zeros inside the unit disk is asymptotically equal to $L/(L+1)$.

In our numerical simulations we generated a large number of $SU(1,1)$ random polynomials of degree N, calculated the zeros using standard techniques, and counted the number of zeros in annuli of fixed width,

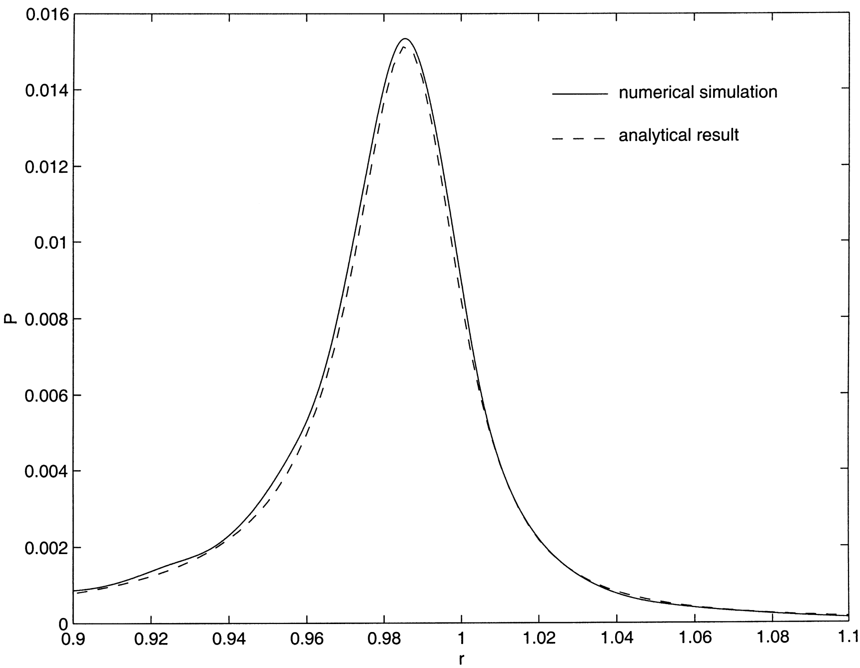

Fig. 2. The scaled density function—theoretical limit and computer simulation.

which are concentrically spread around the origin, per one polynomial. The scaled density was obtained by dividing each of these numbers by the area of the corresponding annulus, as well as by adjusting it with respect to N (according to the previously established result). Figure 2 shows the results of this procedure for $L=4$ and $N=150$, in comparison with the theoretical value as given by Eq. (24).

2.2. *SU*(1, 1) Ensemble of Random Analytic Functions

In this section we consider the $SU(1,1)$ random analytic function,

$$\psi(z)=\sum_{m=0}^{\infty}\sqrt{C_{m+L-1}^{m}}\,a_m z^m, \tag{50}$$

which is obtained from (6) by setting $N=\infty$. Here again C_{m+L-1}^{m} are Newton's binomial coefficients, and a_m are independent standard complex Gaussian random variables.

Proposition 2.2. Random series (50) converges almost surely for all $|z|<1$. For all mutually distinct $z_1,\dots, z_n$, in the disk $|z|<1$ the covariance matrix

$$A_n = (E(\psi(z_p)\,\overline{\psi(z_{p'})}))_{p,\,p'=1,\dots,n} \tag{51}$$

is positive definite.

Proof. *Almost sure convergence.* Observe that $(m+j)/m \leqslant j+1$ hence

$$\frac{(m+1)\cdots(m+L-1)}{m^{L-1}} \leqslant L! \tag{52}$$

and

$$C^m_{m+L-1} = \frac{(m+L-1)\cdots(m+1)}{(L-1)!} \leqslant Lm^{L-1}. \tag{53}$$

Consider the set

$$\Lambda_N = \{a=(a_0, a_1,\dots) : |a_m| \leqslant m \ \ \forall m \geqslant N\}. \tag{54}$$

Then for all $a \in \Lambda_N$ series (50) converges and

$$\lim_{N\to\infty} \text{Prob}\, \Lambda_N = 1, \tag{55}$$

which implies the almost sure convergence.

Positive definiteness. Consider the quadratic form

$$A_n(\mu) = \sum_{p,\,p'=1}^{n} E(\psi(z_p)\,\overline{\psi(z_{p'})})\;\mu_p\,\overline{\mu_{p'}} = E\left|\sum_{p=1}^{n} \mu_p\psi(z_p)\right|^2. \tag{56}$$

Assume that A_n is not positive definite. Then, for some nonzero vector $\mu=(\mu_1,\dots, \mu_n)$,

$$\sum_{p=1}^{n} \mu_p\psi(z_p) = 0 \tag{57}$$

for almost all ψ. Hence, (57) holds for almost ψ with coefficients vector $a \in \Lambda_N$. Assume that $N>n$. There exists a polynomial

$$\psi_0(z) = \sum_{m=0}^{n-1} \sqrt{C^m_{m+L-1}}\; a^0_m\, z^m, \tag{58}$$

such that

$$\psi_0(z_p) = \overline{\mu_p}, \tag{59}$$

so that

$$\sum_{p=1}^{n} \mu_p \psi_0(z_p) = \sum_{p=1}^{n} |\mu_p|^2 \neq 0. \tag{60}$$

Let $a^0 = (a_0^0, a_1^0, \ldots, a_{n-1}^0, 0, 0, \ldots)$. From (54), it is obvious that if $a \in \Lambda_N$ then $(a + a^0) \in \Lambda_N$. The shift $a \to a + a^0$ is a measurable one-to-one transformation in Λ_N, hence for almost all vectors $a \in \Lambda_N$ we have that (57) holds along with

$$\sum_{p=1}^{n} \mu_p [\psi(z_p) + \psi_0(z_p)] = 0. \tag{61}$$

But this implies that

$$\sum_{p=1}^{n} \mu_p \psi_0(z_p) = 0, \tag{62}$$

which is in contradiction with (60). This contradiction proves the positive definiteness of A_n. Proposition 2.2 is proved.

Remark. The above proof gives also the positive definiteness of the covariance matrix A_n corresponding to the random polynomial (6), provided $N > n$.

In this section, we will be interested in the density function $\rho(z)$ of the distribution of zeros of (50) in the disk $|z| < 1$. We have from (8) that

$$E(\psi(z)\, \overline{\psi(z')}) = \sum_{m=0}^{\infty} C_{m+L-1}^{m} (z\overline{z'})^m = \frac{1}{(1 - z\overline{z'})^L}. \tag{63}$$

In particular,

$$E(\psi(z)\, \overline{\psi(z)}) = f(x) \equiv \frac{1}{(1-x)^L}, \qquad x \equiv z\bar{z} = |z|^2. \tag{64}$$

In terms of f the density function $\rho(z)$ of the zeros of (50) is given by:

$$\rho(z) = \frac{1}{\pi}\left[x \left(\frac{f'(x)}{f(x)} \right)' + \frac{f'(x)}{f(x)} \right]. \tag{65}$$

Substituting (64) into this formula gives that

$$\rho(z) = \frac{1}{\pi}\left[x \left(\frac{L}{1-x} \right)' + \frac{L}{1-x} \right] = \frac{L}{\pi(1-x)^2}, \tag{66}$$

or going back to the complex variable z, we obtain the result of Leboeuf,[8]

$$\rho(z) = \frac{L}{\pi(1-|z|^2)^2}. \tag{67}$$

Numerical simulations were similar to those of the previous section. The density was simply obtained by dividing the number of counted zeros by the area of the corresponding annulus. Figure 3 shows the results of this procedure for the degree N equal to 50, 75, 100, and 150, in comparison with the theoretical value as given by Eq. (67).

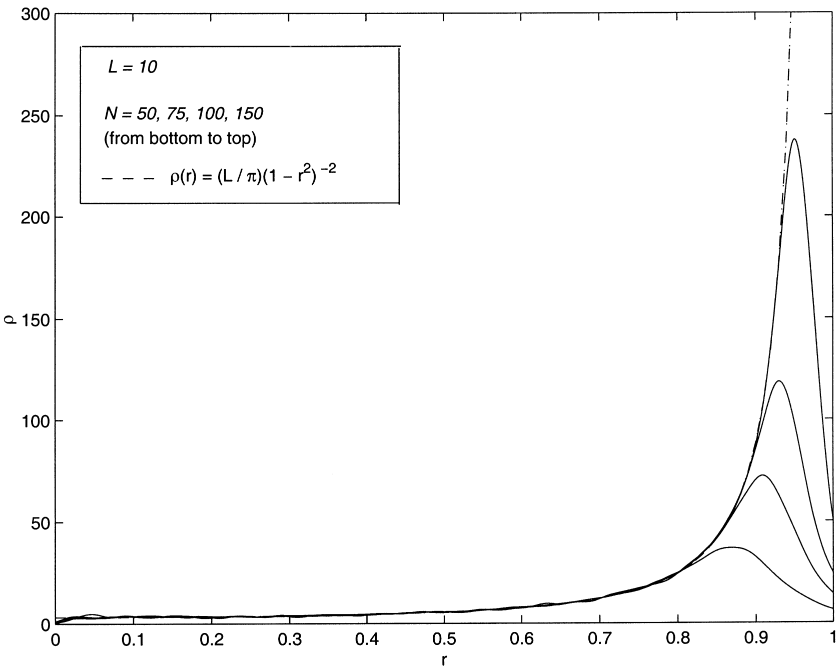

Fig. 3. The unscaled density function—theoretical limit and computer simulation.

3. CORRELATIONS BETWEEN ZEROS

3.1. The *SU*(1, 1) Symmetry

We have the following important property:

Theorem 3.1. The distribution of zeros of the random analytic function (50) is invariant with respect to the action of the group $SU(1, 1)$,

$$z \to \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1). \tag{68}$$

Remark. This implies that all joint distribution functions of zeros of $\psi(z)$ are $SU(1, 1)$ invariant.

Proof. Consider the following homogeneous analytic function of two variables:

$$\Psi(z_0, z_1) = z_0^{-L} \sum_{m=0}^{\infty} \sqrt{C_{m+L-1}^m}\, a_m \left(\frac{z_1}{z_0}\right)^m, \qquad |z_1| < |z_0|. \tag{69}$$

Then

$$\Psi(1, z_1) = \psi(z_1). \tag{70}$$

Let us find the covariance function of $\Psi(z_0, z_1)$:

$$\begin{aligned} E(\Psi(z_0, z_1)\, \overline{\Psi(z_0', z_1')}) &= (z_0\overline{z_0'})^{-L} \sum_{m=0}^{\infty} C_{m+L-1}^m \left(\frac{z_1\overline{z_1'}}{z_0\overline{z_0'}}\right)^m \\ &= (z_0\overline{z_0'})^{-L} \left(1 - \frac{z_1\overline{z_1'}}{z_0\overline{z_0'}}\right)^{-L} \\ &= (z_0\overline{z_0'} - z_1\overline{z_1'})^{-L}. \end{aligned} \tag{71}$$

The action of a matrix $A \in SU(1, 1)$ preserves the (1,1) indefinite scalar product $z_0\overline{z_0'} - z_1\overline{z_1'}$. Thus, the covariance function of the Gaussian random analytic function $\Psi(z_0, z_1)$ is $SU(1, 1)$ invariant. In addition,

$$E\Psi(z_0, z_1) = 0, \qquad E\Psi(z_0, z_1)\, \Psi(z_0', z_1') = 0. \tag{72}$$

For the Gaussian random function, the first two moments determine it uniquely. This implies that the distribution of zeros of $\Psi(z_0, z_1)$ is $SU(1, 1)$ invariant. Restricting this to $z_0 = 1$ we get that the distribution of zeros of $\psi(z)$ is $SU(1, 1)$ invariant as well. Theorem 3.1 is proved.

Recall that a general formula for a matrix $A \in SU(1, 1)$ is the following:

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad |b|^2 - |a|^2 = 1, \tag{73}$$

so that it depends on three real parameters. An $SU(1, 1)$-invariant metric is

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, \tag{74}$$

and the corresponding distance $\tau(z_1, z_2)$ on the disk $\{|z| < 1\}$ is determined by the equation

$$\tanh(\tau/2) = \frac{|z_1 - z_2|}{|1 - z_1 \overline{z_2}|}. \tag{75}$$

The corresponding $SU(1, 1)$-invariant volume element is

$$\frac{4dx\, dy}{(1 - x^2 - y^2)^2}. \tag{76}$$

Theorem 3.1 implies the $SU(1, 1)$ invariance for the normalized correlation functions.

3.2. Correlation Functions—Preliminaries

The n-point correlation function $K_n(z_1,..., z_n)$ is determined by the condition that for any test functions $\varphi_1(z),..., \varphi_n(z)$, which are infinitely differentiable and compactly supported in the disk $\{|z| < 1\}$, such that their supports do not intersect,

$$E \prod_{p=1}^{n} \left(\sum_{j=1}^{\infty} \varphi_p(\zeta_j) \right) = \int_{\mathbb{C}^n} K_n(z_1,..., z_n) \prod_{p=1}^{n} [\varphi(z_p)\, dz_p], \tag{77}$$

where ζ_j are zeros of the $SU(1, 1)$ random analytical function $\psi(z)$ (we denote them by ζ_j to distinguish them from the variables z_j). The sum over j is, in fact, finite because $\varphi_p(z)$ has a compact support. The general formula for $K_n(z_1,..., z_n)$ is given by the Kac–Rice expression

$$K_n(z) = \int d\xi\, D_n(0, \xi; z) \prod_{p=1}^{n} (\xi_p \xi_p^*), \qquad z = (z_1,..., z_n), \quad \xi = (\xi_1,..., \xi_n), \tag{78}$$

where $D_n(x, \xi; z)$, $x, \xi \in C^n$, is the distribution density of the two random vectors

$$X = (\psi(z_1),..., \psi(z_n)), \qquad \Xi = (\psi'(z_1),..., \psi'(z_n)). \tag{79}$$

Formula (78) is derived in ref. 2 in a much more general situation of sections of powers of a line bundle over a complex manifold, as a generalization of the original formula by Kac[20] and Rice[21] (see also refs. 1 and 18).

For the Gaussian random functions, formula (78) can be specialized as follows (see refs. 1, 2, and 18):

$$K_n(z) = \frac{1}{\pi^n \det A_n} \left\langle \prod_{p=1}^{n} (\xi_p \xi_p^*) \right\rangle_{\Lambda_n}, \tag{80}$$

where

$$A_n = (E\, \psi(z_p)\, \overline{\psi(z_{p'})})_{p,\, p'=1,...,n}, \tag{81}$$

and $\langle \cdot \rangle_{\Lambda_n}$ stands for averaging with respect to Gaussian complex random variables $\xi_1,..., \xi_n$, such that

$$(E\, \xi_p \overline{\xi_{p'}})_{p,\, p'=1,...,n} = \Lambda_n, \qquad E\, \xi_p = 0, \quad E\, \xi_p \xi_{p'} = 0, \quad p, p' = 1,..., n, \tag{82}$$

where

$$\Lambda_n = C_n - B_n^* A_n^{-1} B_n \tag{83}$$

and

$$B_n = (E\, \psi(z_p)\, \overline{\psi'(z_{p'})})_{p,\, p'=1,...,n}, \qquad C_n = (E\, \psi'(z_p)\, \overline{\psi'(z_{p'})})_{p,\, p'=1,...,n}. \tag{84}$$

For $n = 1$, formula (80) reduces to

$$K_1(z) = \frac{1}{\pi} \frac{A_n(z)\, C_n(z) - B_n(z)\, \overline{B_n(z)}}{A_n^2(z)}, \tag{85}$$

where

$$A_n(z) = E\, \psi(z)\, \overline{\psi(z)}, \qquad B_n(z) = \frac{\partial A_n(z)}{\partial z}, \qquad C_n(z) = \frac{\partial^2 A_n(z)}{\partial z \partial \bar{z}}. \tag{86}$$

The 1-point correlation function $K_1(z)$ is nothing else than the density function of the distribution of zeros and formula (85) is equivalent to the Poincaré–Lelong type formula (13).

The normalized correlation function $k_n(z_1,..., z_n)$ is defined as

$$k_n(z_1,..., z_n) = \frac{K_n(z_1,..., z_n)}{K_1(z_1)\cdots K_1(z_n)}. \tag{87}$$

It satisfies the following theorem.

Theorem 3.2. The function $k_n(z_1,..., z_n)$ is invariant with respect to the action of the group $SU(1, 1)$,

$$k_n(Az_1,..., Az_n) = k_n(z_1,..., z_n), \qquad \forall A \in SU(1, 1), \tag{88}$$

where

$$Az = \frac{az+b}{cz+d}. \tag{89}$$

Remark. This implies that $k_n(z_1,..., z_n)$ is a function of pairwise distances $\tau(z_p, z_q)$ defined by (74).

Proof. By Theorem 3.1, the distribution $K_n(z_1,..., z_n)\, dz_1 \cdots dz_n$ is $SU(1, 1)$ invariant. Also, the distribution $K_1(z_1)\cdots K_1(z_n)\, dz_1 \cdots dz_n$ is $SU(1, 1)$ invariant. Hence their quotient (the Radon–Nikodim derivative),

$$\frac{K_n(z_1,..., z_n)\, dz_1 \cdots dz_n}{K_1(z_1)\cdots K_1(z_n)\, dz_1 \cdots dz_n} = k_n(z_1,..., z_n) \tag{90}$$

is $SU(1, 1)$ invariant, which was stated. Theorem 3.2 is proved.

The normalized correlation function can be expressed as a supersymmetric (Berezin) integral,

$$k_n(z_1,..., z_n) = \frac{1}{\det A_n} \int \frac{1}{\det[I + \Lambda_n(z)\, \Omega]}\, d\eta, \tag{91}$$

where Ω is $n \times n$ matrix,

$$\Omega = (\delta_{pp'} \eta_{p'} \bar{\eta}_p)_{p,\, p' = 1,..., n}, \tag{92}$$

and the $\eta_p, \bar{\eta}_p$ are anti-commuting (fermionic) variables, with $d\eta = \prod_p d\bar{\eta}_p d\eta_p$. The integral in (91) is a Berezin integral, which is evaluated by simply taking the coefficient of the top degree form of the integrand $\frac{1}{\det[I+\Lambda_n(z)\,\Omega]}$. For a derivation of formula (91), see ref. 14.

3.3. The Two-Point Correlation Function

From (50), (67), (81), and (84), we can directly obtain the following expressions for A_n, B_n, and C_n in the case of a two-point correlation function $k_2(z_1, z_2)$:

$$A_2 = \begin{pmatrix} s_{11}{}^{-L} & s_{12}{}^{-L} \\ s_{21}{}^{-L} & s_{22}{}^{-L} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} Lz_1 s_{11}{}^{-L-1} & Lz_1 s_{12}{}^{-L-1} \\ Lz_2 s_{21}{}^{-L-1} & Lz_2 s_{22}{}^{-L-1} \end{pmatrix} \tag{93}$$

$$C_2 = \begin{pmatrix} L(1+L|z_1|^2)\, s_{11}{}^{-L-2} & L(1+Lz_1\overline{z_2})\, s_{12}{}^{-L-2} \\ L(1+L\overline{z_1}z_2)\, s_{21}{}^{-L-2} & L(1+L|z_2|^2)\, s_{22}{}^{-L-2} \end{pmatrix},$$

with $s_{11} = 1-|z_1|^2$, $s_{12} = 1-z_1\overline{z_2}$, $s_{21} = 1-\overline{z_1}z_2$, and $s_{22} = 1-|z_2|^2$. By Theorem 3.2, $k_2(z_1, z_2)$ is $SU(1, 1)$ invariant, that is

$$k_2(Az_1, Az_2) = k_2(z_1, z_2), \qquad \forall A \in SU(1, 1). \tag{94}$$

The action of $SU(1, 1)$ is transitive, hence we can move z_1 to the origin, and by rotation, we can then move z_2 to the positive half-axis. Therefore, we will assume that $z_1 = 0$ and $z_2 = r > 0$. In that case, Eqs. (93) become

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & (1-r^2)^{-L} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ Lr & Lr(1-r^2)^{-L-1} \end{pmatrix} \tag{95}$$

$$C_2 = \begin{pmatrix} L & L \\ L & L(1+Lr^2)(1-r^2)^{-L-2} \end{pmatrix}.$$

From (95) and (83), we further obtain

$$\Lambda_2 = \begin{pmatrix} L-L^2r^2\,\frac{1}{b} & L-L^2r^2\,\frac{a}{b} \\ L-L^2r^2\,\frac{a}{b} & L(1+Lr^2)(1-r^2)^{-L-2}-L^2r^2\,\frac{a^2}{b} \end{pmatrix}, \tag{96}$$

where $a=(1-r^2)^{-L-1}$ and $b=(1-r^2)^{-L}-1$. For the two-point case, formula (80) reduces to the following expression:

$$K_2(z_1, z_2)=\frac{1}{\pi^2 \det A_2}(A_{2,11}A_{2,22}+A_{2,12}A_{2,21}), \tag{97}$$

which, when combined with (87), (85), and (67), results in the following formula for the normalized two-point correlation function:

$$k_2(z_1, z_2)=[(1-r^2)^{3L+2}+((L^2-2L-2)\,r^4+(4L+4)\,r^2-1)(1-r^2)^{2L} \\ +((L+1)^2r^4-(4L+2)\,r^2-1)(1-r^2)^L+1]/(1-(1-r^2)^L)^3, \tag{98}$$

where according to (75),

$$r=\tanh\left(\frac{\tau}{2}\right)=\frac{|z_1-z_2|}{|1-z_1\overline{z_2}|}. \tag{99}$$

When $L=1$, formula (98) simplifies to

$$k_2(z_1, z_2)=r^2(2-r^2), \qquad L=1. \tag{100}$$

Plots of $k_2(\tanh(\frac{\tau}{2}))$ for $L=1$, 5 and 50 are shown in Fig. 4.

As it can be seen from the plot, the two-point correlation function goes to 0 as $\tau\to 0$ (or, in other words, as $r\to 0$). The limiting behavior can be obtained through a series expansion of (98). The following expression was obtained using Maple™:

$$k_2(r)=\frac{1}{2}\frac{(L+1)^2}{L}r^2-\frac{1}{4}\frac{(L+1)^2}{L}r^4-\frac{1}{36}\frac{(L^2-1)^2}{L}r^6 \\ -\frac{1}{72}\frac{(L^2-1)^2}{L}r^8+O(r^{10}), \tag{101}$$

which shows dominating quadratic behavior in the neighborhood of $r=0$. Thus, there is a quadratic repulsion between zeros.

We are also interested in the asymptotic behavior of the correlation function as $L\to\infty$. For this purpose, we introduce the scaling

$$r=\frac{u}{\sqrt{L}}. \tag{102}$$

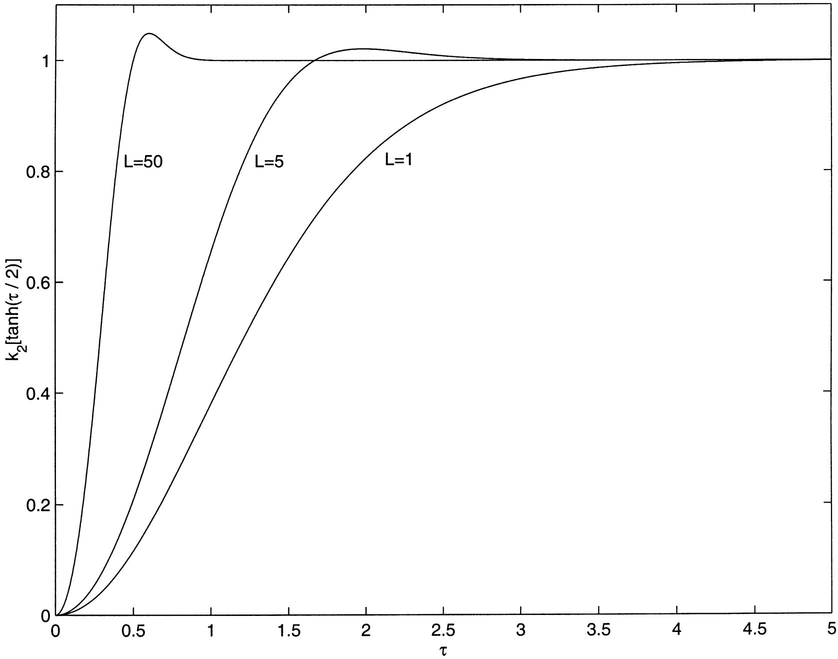

Fig. 4. The two-point correlation function.

Substituting this into (98) and taking the limit $L \to \infty$, we obtain the following expression:

$$k_2(u) = \frac{e^{3u^2} + (u^4 - 4u^2 - 1)\, e^{2u^2} + (u^4 + 4u^2 - 1)\, e^{u^2} + 1}{(e^{u^2} - 1)^3}, \tag{103}$$

which can also be written as

$$k_2(u) = \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}, \qquad t = \frac{u^2}{2}, \tag{104}$$

and agrees with the result obtained by Hannay.[1] A plot of (104) is shown in Fig. 5.

In fact, under the scaling (102), the $SU(1, 1)$ random analytic function converges, as $L \to \infty$, to the W_1 random analytic function (cf. ref. 8). Indeed,

$$\psi\left(\frac{u}{\sqrt{L}}\right) = \sum_{m=0}^{\infty} \sqrt{\frac{L(L+1)\cdots(L+m-1)}{L^m m!}}\, a_m u^m. \tag{105}$$

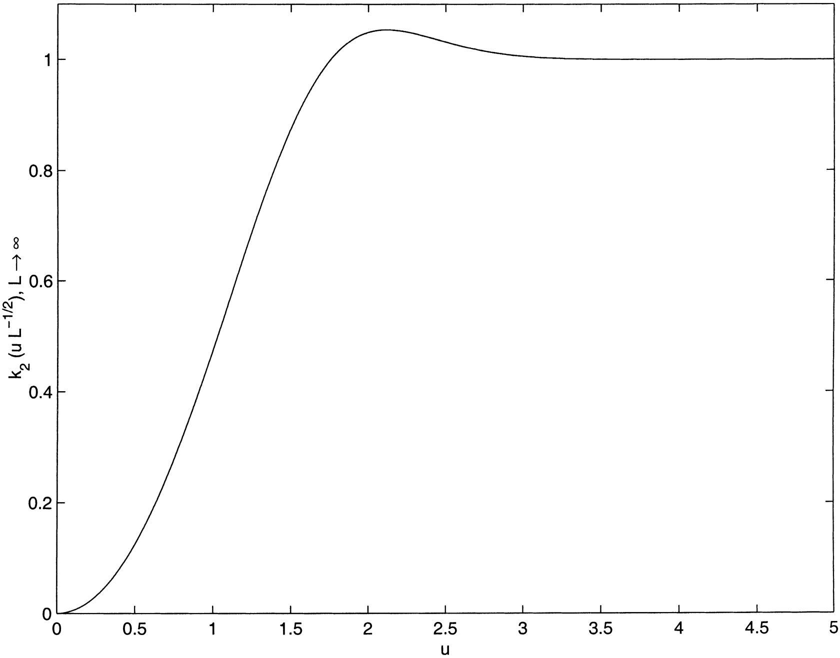

Fig. 5. Asymptotics of the two-point correlation function as $L \to \infty$.

As $L \to \infty$, the expression under the radical approaches $1/m!$, so that $\psi(u/\sqrt{L})$ approaches

$$\psi_0(u) = \sum_{m=0}^{\infty} \sqrt{\frac{1}{m!}}\, a_m u^m, \tag{106}$$

which is the W_1 random analytic function.

APPENDIX. CORRELATIONS BETWEEN INNER AND OUTER ZEROS

In this Appendix, we consider the limit of correlations between zeros of the random polynomial (6) as $N \to \infty$. In the open disk $\{|z| < 1\}$, the random polynomial (6) approaches the $SU(1,1)$ random analytic function (50), and the correlations between zeros of the random polynomial approach the ones of the random analytic function. However, according to formula (49), there is a $1/(L+1)$ fraction of the zeros outside of the unit disk. The limiting correlations between those, outer zeros, and between inner and outer zeros are described by the following theorem. Let

$k_n^L(z_1,\ldots, z_n)$ denote the normalized n-point correlation function (87) corresponding to the parameter L. Indication of L is important for the theorem. Let furthermore $k_{nN}^L(z_1,\ldots, z_n)$ denote the normalized n-point correlation function for the zeros of polynomial (6).

Theorem A.1. Assume that $|z_1|,\ldots, |z_m| < 1$ and $|z_{m+1}|,\ldots, |z_n| > 1$. Then

$$\lim_{N\to\infty} k_{nN}^L(z_1,\ldots, z_n) = k_m^L(z_1,\ldots, z_m)\, k_{n-m}^1(z_{m+1}^{-1},\ldots, z_n^{-1}), \tag{107}$$

so that in the limit $N \to \infty$, the inner and outer zeros become independent, and the limiting correlations between the outer zeros coincide, after the change of variable $z \to 1/z$ in the argument, with the correlations between zeros of the $SU(1, 1)$ random analytic function with the parameter $L = 1$.

Proof. We will first find correlations between the outer zeros and then prove the independence of the inner and outer zeros.

Correlations between outer zeros. Consider the random polynomial (6) and define another random polynomial,

$$\varphi(z) = \frac{1}{\sqrt{C_{N+L-1}^N}}\, z^N \psi(z^{-1}). \tag{108}$$

Observe that if z_j, $|z_j| > 1$, is a zero of $\psi(z)$ then z_j^{-1} is a zero of $\varphi(z)$ and $|z_j^{-1}| < 1$. Consider $\varphi(z)$ in the disk $|z| < 1$. From (6),

$$\varphi(z) = a_N + \sqrt{\frac{N\cdots(N+L-2)}{(N+1)\cdots(N+L-1)}}\, a_{N-1} z + \sqrt{\frac{(N-1)\cdots(N+L-3)}{(N+1)\cdots(N+L-1)}}\, a_{N-2} z^2 + \cdots. \tag{109}$$

As $N \to \infty$, the expressions under the radicals approach 1 from below. In addition, we can replace a_{N-m} by a_m, because they are the same standard random variables. Thus, as $N \to \infty$, $\varphi(z)$ approaches the random function

$$\varphi(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad |z| < 1, \tag{110}$$

which is the $SU(1, 1)$ random analytic function with the parameter $L = 1$. Observe that $\varphi(z)$ is a Gaussian random polynomial and for its correlations we have formula (80). From this formula, we obtain the convergence of correlations between zeros of $\varphi(z)$ as $N \to \infty$ to the ones of the $SU(1, 1)$ random analytic function with $L = 1$.

Independence. We introduce the random function $\eta(z)$ such that $\eta(z) = \psi(z)$ for $|z| \leqslant 1$ and

$$\eta(z) = \frac{1}{\sqrt{C^N_{N+L-1}}} z^{-N} \psi(z), \qquad |z| > 1. \tag{111}$$

Then the zeros of $\eta(z)$ and $\psi(z)$ coincide. In addition, $\eta(z)$ is a Gaussian random field and its covariance function $E(\eta(z)\,\overline{\eta(z')})$ coincides with $E(\psi(z)\,\overline{\psi(z')})$ when $|z|, |z'| < 1$, and hence, as $N \to \infty$, it approaches the correlation function (63) of the $SU(1,1)$ random analytic function with the parameter L. Similarly, as we saw above in this appendix, when $|z|, |z'| > 1$, $E(\eta(z)\,\overline{\eta(z')})$ approaches the $SU(1,1)$ covariance function with $L = 1$, if we replace z, z' by z^{-1}, z'^{-1}, respectively. Consider now the correlation function $E(\eta(z)\,\overline{\eta(z')})$ when $|z| > 1$ and $|z'| < 1$. From (111),

$$E(\eta(z)\,\overline{\eta(z')}) = \frac{1}{z^N \sqrt{C^N_{N+L-1}}} \sum_{m=0}^{N} C^m_{L+m-1} (z\overline{z'})^m. \tag{112}$$

Assume that $|zz'| \leqslant 1$. Then we can estimate the correlation function as follows:

$$|E(\eta(z)\,\overline{\eta(z')})| \leqslant \frac{1}{|z|^N \sqrt{C^N_{N+L-1}}} \sum_{m=0}^{N} C^m_{L+m-1} \leqslant \frac{N^L}{|z|^N}. \tag{113}$$

If $|zz'| > 1$, then we similarly obtain that

$$|E(\eta(z)\,\overline{\eta(z')})| = \left| \frac{z'^N}{\sqrt{C^N_{N+L-1}}} \sum_{m=0}^{N} C^m_{L+m-1} (z\overline{z'})^{m-N} \right| \leqslant N^L\, |z'|^N. \tag{114}$$

Combining the two cases we can write

$$|E(\eta(z)\,\overline{\eta(z')})| \leqslant N^L (\max\{|z|^{-1}, |z'|\})^N. \tag{115}$$

This shows that the values of $\eta(z)$ inside and outside of the unit disk become independent as $N \to \infty$. Hence their zeros become independent. Explicit estimates for the correlations between the inner and outer zeros follow from formula (80) applied to $\eta(z)$.

ACKNOWLEDGMENTS

Research of P.B. is partially supported by NSF Grant #DMS-9970625 and this support is gratefully acknowledged.

REFERENCES

1. J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state, *J. Phys. A* **29**:101–105 (1996).
2. P. Bleher, B. Shiffman, and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Invent. Math.* **142**:351–395 (2000).
3. E. Bogomolny, O. Bohigas, and P. Leboeuf, Quantum chaotic dynamics and random polynomials, *J. Statist. Phys.* **85**:639–679 (1996).
4. P. Leboeuf and P. Shukla, Universal fluctuations of zeros of chaotic wavefunctions, *J. Phys. A* **29**:4827–4835 (1996).
5. H. J. Korsch, C. Miller, and H. Wiescher, On the zeros of the Husimi distribution, *J. Phys. A* **30**:L677–L684 (1997).
6. S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, *J. Statist. Phys.* **92**:431–518 (1998).
7. P. J. Forrester and G. Honner, Exact statistical properties of the zeros of complex random polynomials, *J. Phys. A* **32**:2961–2981 (1999).
8. P. Leboeuf, Random analytic chaotic eigenstates, *J. Statist. Phys.* **95**:651–664 (1999).
9. B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, *Commun. Math. Phys.* **200**:661–683 (1999).
10. M. L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, Boston, 1991).
11. M. Shub and S. Smale, Complexity of Bezout's theorem II: Volumes and probabilities, in *Computational Algebraic Geometry (Nice, 1992)*, Progr. Math. 109, Birkhäuser, Boston (1993), pp. 267–285.
12. P. Bleher, B. Shiffman, and S. Zelditch, Universality and scaling of zeros on symplectic manifolds, in *Random Matrix Models and Their Applications*, Vol. 40, P. Blehr and A. Its, eds. (Cambridge University Press, Cambridge, 2001), pp. 31–70, http://xxx.lanl.gov/abs/math-ph/0002039.
13. P. Bleher, B. Shiffman, and S. Zelditch, Poincaré–Lelong approach to universality and scaling of correlations between zeros, *Commun. Math. Phys.* **208**:771–785 (2000).
14. P. Bleher, B. Shiffman, and S. Zelditch, Correlations between zeros and supersymmetry, http://xxx.lanl.gov/abs/math-ph/0011016 (to appear in *Commun. Math. Phys.*).
15. B. Shiffman and S. Zelditch, Random almost holomorphic sections of ample line bundles on symplectic manifolds, e-print (2000), http://xxx.lanl.gov/abs/math.SG/0001102.
16. B. Jancovici and G. Téllez, Two-dimensional Coulomb systems on a surface of constant negative curvature, *J. Statist. Phys.* **91**:953–977 (1998).
17. K. Zyczkowski and H.-J. Sommers, Truncations of random unitary matrices, *J. Phys. A* **60**:2045–2057 (2000).
18. P. Bleher and X. Di, Correlations between zeros of a random polynomial, *J. Statist. Phys.* **88**:269–305 (1997).
19. A. Bharucha-Reid and M. Sambadham, *Random Polynomials* (Academic Press, New York, 1986).
20. M. Kac, On the average number of real roots of a random algebraic equation, *Bull. Amer. Math. Soc.* **49**:314–320 (1943).
21. S. O. Rice, Mathematical analysis of random noise, *Bell System Tech. J.* **23**:282–332 (1944); **24**:46–156 (1945); reprinted in *Selected Papers on Noise and Stochastic Processes,* (Dover, New York, 1954), pp. 133–294.