Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that

\[ m(A \cap B) = 0.9, \quad m(A \cap C) = 0.9, \quad m(B \cap C) = 0.9, \quad m(A \cup B \cup C) = 1. \]

Prove that

\[ m(A \cap B \cap C) \geq 0.85. \]

Problem 2. Suppose that a sequence $E_1, E_2, \ldots$ of measurable sets on $[0, 1]$ satisfies the Cauchy condition, so that for every $\varepsilon > 0$ there exists $N > 0$ such that

\[ m(E_n \Delta E_m) \leq \varepsilon, \]

for all $n, m \geq N$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that there exists a measurable set $E$ on $[0, 1]$ such that

\[ \lim_{n \to \infty} m(E_n \Delta E) = 0. \]

Problem 3. Evaluate the limit,

\[ \lim_{n \to \infty} \int_0^1 \frac{nx^n dx}{(1 + x^n)^2}, \]

and justify your answer.

Problem 4. Let us enumerate all rational points on $[0, 1]$,

\[ \mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}. \]

Consider the function

\[ f(x) = \sum_{n=1}^{\infty} \frac{|x - r_n|}{n^{3/2}} \]

on $[0, 1]$. Prove that the function $f(x)$ is differentiable almost everywhere.

Problem 5. Prove that if $f_n$ converges to $f$ in $L^p(E)$ and $g_n$ converges to $g$ in $L^q(E)$, where

\[ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad p > 0, \quad q > 0, \quad r \geq 1, \]

then $f_n g_n$ converges to $f g$ in $L^r(E)$.

Problem 6. Let

\[ f_n(x) = \sum_{k=1}^{n} \frac{\sin k x}{k^{2/3}}. \]

Prove that there exists an integrable function $f(x)$ on the interval $[0, 2\pi]$ such that

\[ \lim_{n \to \infty} \int_{[0,2\pi]} |f(x) - f_n(x)| dx = 0. \]