Effective actions at finite temperature and thermodynamic properties of some bosonic field theories

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Frontiers in Casimir Physics

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Effective action at finite temperature. Scalar field

Partition function:

$$Z[\phi] = \int d[\phi] \, e^{-\frac{1}{2}\int d^D x \phi A \phi}$$

Effective action:

$$\Gamma = -\log Z = \log \left(\det A_{PBC}\right)^{-\frac{1}{2}}$$

PBC in Euclidean “time” τ: $\phi(\beta) = \phi(0), \quad \beta = \frac{1}{T}$

Natural units: $\hbar = c = k_B = 1$

Zeta regularization of the determinant:

$$\Gamma \equiv \frac{1}{2} \left. \frac{d}{ds} \right|_{s=0} \sum_k d_k \lambda_k{}^{-s}$$

Conformal scalar field in $M^3 \times S^1$

In this case, the relevant operator is the conformal Laplacian

$$A = \Delta_c = -\Delta^4 + \frac{R}{6},$$

with R the scalar curvature of M^3 and Δ^4 the Laplace-Beltrami Laplacian in four dimensions

For S^3, $R = \frac{6}{a}$, with a the radius of S^3, and

$$\Delta_c = -\Delta + \frac{1}{a} = -\left(\partial_\tau\right)^2 - \Delta^3 + \frac{1}{a}$$

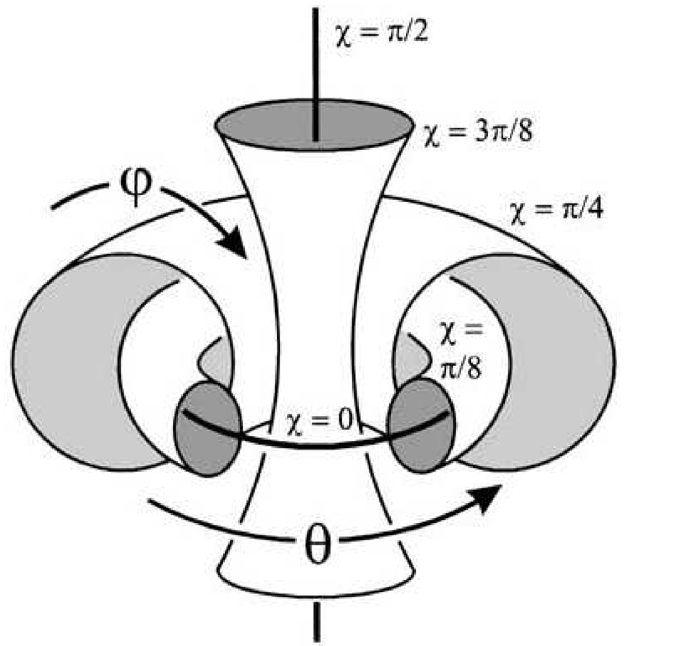
S^3 as a collection of nested tori (solid torus)

$$S^3: \quad x^2 + y^2 + u^2 + v^2 = |z|^2 + |w|^2 = 1$$

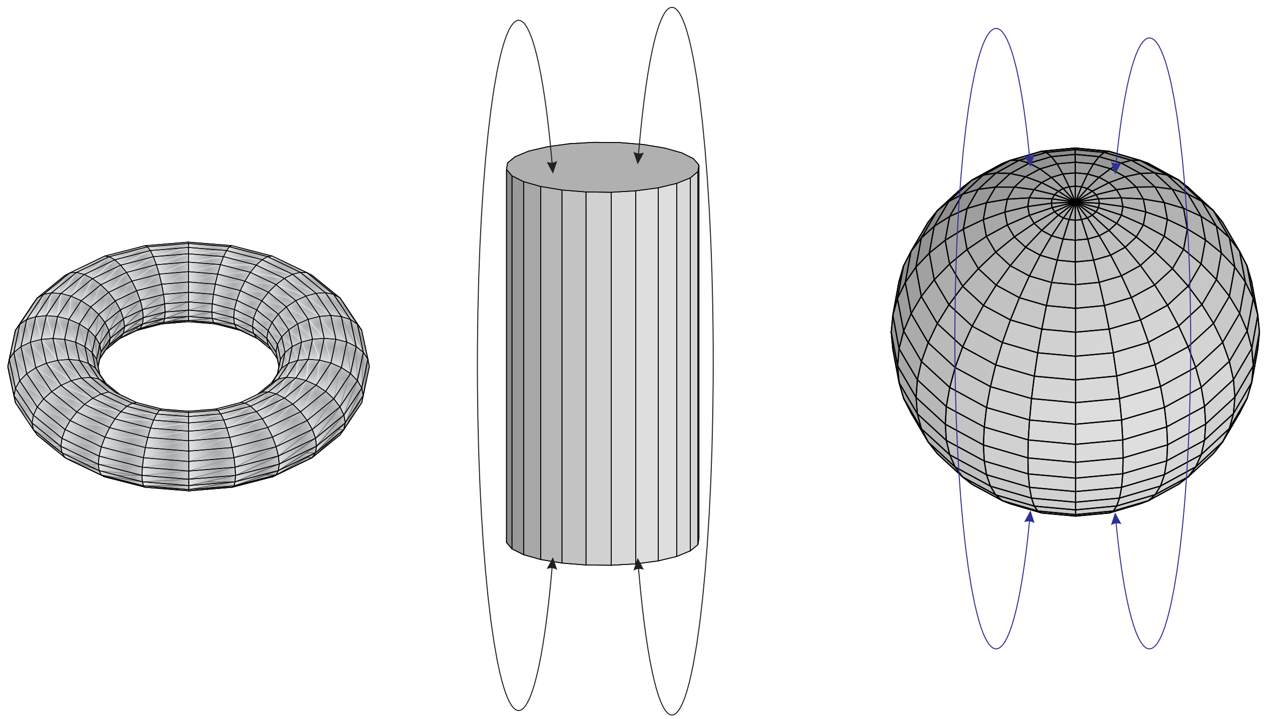
$$z = x + iy = r_1 e^{i\theta}$$
$$w = u + iv = r_2 e^{i\varphi}$$

$$r_1{}^2 = 1 - r_2{}^2$$

Degenerate cases (circles): $r_2 = 0$ and $r_2 = 1$



$$
\begin{aligned}
x &= \cos\chi\cos\theta \\
y &= \cos\chi\sin\theta \qquad 0 < \varphi,\ \theta \leq 2\pi \\
u &= \sin\chi\cos\varphi \qquad 0 \leq \chi \leq \frac{\pi}{2} \\
v &= \sin\chi\sin\varphi
\end{aligned}
$$



Effective action $S^3 \times S^1$

$\lambda_{k,l} = (2\pi a l/\beta)^2 + k^2,\ k = 1, ..., \infty,\ l = -\infty, ...\infty$ and $d_k = k^2$

$$\zeta_{S^3}(s) = \mu^{2s} \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} k^2 \left((2\pi l/\beta)^2 + \left(\frac{k}{a}\right)^2 \right)^{-s}$$

Low temperatures ($\beta \to \infty$)

$$\zeta_{S^3}(s) = \left(\frac{2\pi}{\mu\beta}\right)^{-2s} \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} k^2 \left[\left(\frac{k\beta}{2\pi a}\right)^2 + l^2 \right]^{-s}$$

$$\delta^{-1} = \frac{\beta}{2\pi a} \to \infty \quad \text{as} \quad \beta \to \infty$$

Mellin transform

$$\zeta_{S^3}(s) = \frac{1}{\Gamma(s)} \left(\frac{2\pi}{\mu\beta}\right)^{-2s} \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} k^2 \int_0^\infty dt\, t^{s-1} e^{-\left[\left(\frac{k\beta}{2\pi a}\right)^2 + l^2\right] t}$$

To perform analytic extension, use

$$\sum_{l=-\infty}^{\infty} e^{-tl^2} = \left(\frac{\pi}{t}\right)^{\frac{1}{2}} \sum_{l=-\infty}^{\infty} e^{-\frac{\pi^2 l^2}{t}}$$

$$\zeta_{S^3}(s) = \frac{\pi^{1/2}}{\Gamma(s)} \left(\frac{2\pi}{\mu\beta}\right)^{-2s} \sum_{k=1}^{\infty} k^2 \left\{ \int_0^\infty dt\, t^{s-\frac{3}{2}} e^{-\left(\frac{k\beta}{2\pi a}\right)^2 t} + \right.$$

$$\left. +2 \sum_{l=1}^{\infty} \int_0^\infty dt\, t^{s-\frac{3}{2}} e^{\left(\frac{k\beta}{2\pi a}\right)^2 t} e^{-\frac{\pi^2 l^2}{t}} \right\}$$

$$\zeta_{S^3}^{l=0}(s) = s\,\frac{\beta\mu^{2s}}{2\pi^{1/2}}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)}\sum_{k=1}^{\infty}\frac{k^{3-2s}}{a^{1-2s}}$$

$$\zeta_{S^3}^{l\neq 0}(s) = s\,\frac{2\pi^{1/2}}{\Gamma(s+1)}\left(\frac{2\pi}{\mu\beta}\right)^{-2s}\sum_{k=0}^{\infty}k^2\sum_{l=1}^{\infty}2\left(\frac{2\pi^2 al}{k\beta}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(\frac{k\beta l}{a}\right)$$

Note: $\zeta_{S^3}(0) = 0$ Derivative w.r. to s easy to evaluate. No dependence on μ remaining:

$$S_{\mathrm{eff},\,\mathrm{S}^3} = \frac{\beta}{2a}\zeta_R(-3) - 2\pi^{1/2}\sum_{k=1}^{\infty}k^2\sum_{l=1}^{\infty}\left(\frac{2\pi^2 al}{k\beta}\right)^{-1/2}K_{-\frac{1}{2}}\left(\frac{k\beta l}{a}\right)$$

$$S_{\mathrm{eff},\,\mathrm{S}^3} = \frac{\beta}{2a}\zeta_R(-3) - \sum_{k=1}^{\infty}k^2\sum_{l=1}^{\infty}l^{-1}e^{-\frac{k\beta l}{a}}$$

$$S_{\mathrm{eff},\,\mathrm{S}^3} = \frac{\beta}{2a}\zeta_R(-3) + \sum_{k=1}^{\infty}k^2\log\left(1 - e^{-k\beta/a}\right)$$

High temperature ($\beta \to 0$)

Back to

$$\zeta_{S^3}(s) = \mu^{2s} \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} k^2 \left((2\pi l/\beta)^2 + \left(\frac{k}{a}\right)^2 \right)^{-s}$$

$$\zeta_{S^3}(s) = (\mu\, a)^{2s} \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} k^2 \left((2\pi a l/\beta)^2 + k^2 \right)^{-s} = \zeta_{l=0}(s) + \zeta_{l\neq 0}(s)$$

$$\zeta_{l=0}(s) = (\mu\, a)^{2s} \sum_{k=1}^{\infty} k^{-(2s-2)} = (\mu\, a)^{2s} \zeta_R(2s-2)$$

$$\begin{aligned} \zeta_{l\neq 0}(s) &= (\mu\, a)^{2s} \left\{ \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \left[k^2 + \left(\frac{2l\pi a}{\beta} \right)^2 \right]^{-(s-1)} \right. \\ &- \left. \sum_{l=1}^{\infty} \left(\frac{2l\pi a}{\beta} \right)^2 \sum_{k=-\infty}^{\infty} \left[k^2 + \left(\frac{2l\pi a}{\beta} \right)^2 \right]^{-s} \right\} \end{aligned}$$

$$\zeta_{l\neq 0}(s) = (\mu\, a)^{2s} \left\{ \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \left[k^2 + \left(\frac{2l\pi a}{\beta}\right)^2 \right]^{-(s-1)} - \sum_{l=1}^{\infty} \left(\frac{2l\pi a}{\beta}\right)^2 \sum_{k=-\infty}^{\infty} \left[k^2 + \left(\frac{2l\pi a}{\beta}\right)^2 \right]^{-s} \right\}$$

Same procedure as before: Mellin transform. This time, invert the Jacobi theta function defined by k-sum

$$S_{\text{eff},\, \text{S}^3} = \frac{1}{4\pi^2}\zeta_R(3) - \frac{8\pi^4}{3}\left(\frac{a}{\beta}\right)^3 \zeta_R(-3) + \frac{1}{4\pi^2}\sum_{k,l=1}^{\infty}\frac{1}{k^3}\left[2 + 2\frac{4\pi^2 kal}{\beta} + \left(\frac{4\pi^2 kal}{\beta}\right)^2\right] e^{-\frac{4\pi^2 kal}{\beta}}$$

Note: dependence on δ, instead of δ^{-1}. No evident inversion symmetry

Thermodynamic properties in S^3

$$F = \frac{1}{\beta} S_{\text{eff}} \qquad E = \frac{\partial}{\partial\beta} S_{\text{eff}} \qquad S = \beta(E - F)$$

Zero temperature limit ($\beta \to \infty$)

$$F = E = \frac{1}{240a} \qquad S = 0 \text{ (Third law OK)}$$

Infinite temperature limit ($\beta \to 0$)

$$F = \frac{\zeta_R(3)}{4\pi^2\beta} - \frac{\pi^4 a^3}{45\beta^4} \qquad E = \frac{\pi^4 a^3}{15\beta^4} \qquad S = \frac{4\pi^4 a^3}{45\beta^3} - \frac{\zeta_R(3)}{4\pi^2}$$

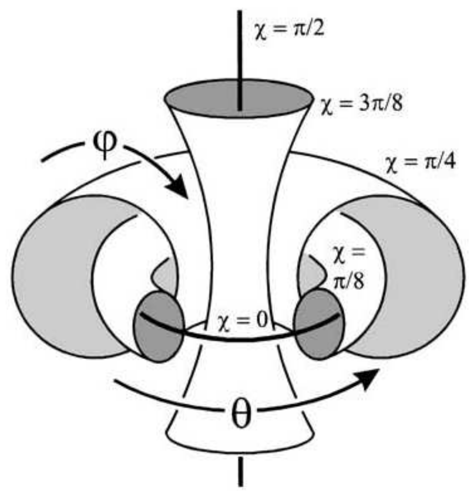
Planck+ topological entropy

Only inversion invariance: $\delta^2(aE)(\delta^{-1}) = \delta^{-2}(aE)(\delta) \qquad \delta = \frac{2\pi a}{\beta}$

Valid at arbitrary temperature - Similar to Ravndall-Tollefsen's symmetry in thermal Casimir effect

General discussion in J.S. Dowker, "Zero modes, entropy bounds and partition functions", Class.Quant.Grav.20:L105-L114 (2003)

Homogeneous lens spaces $L(p, 1)$



$$\theta \to \theta + \frac{2\pi}{p}, \quad \varphi \to \varphi + \frac{2\pi}{p}, \quad \chi \to \chi$$

Preserves $ds^2 = d\chi^2 + \cos^2\chi d\theta^2 + \sin^2\chi d\varphi^2$ (isometry + acts freely)

S^3/Z_p: identify those angles. Get a manifold $L(p, 1)$

Spectrum does not change . Degeneracies do

$$\lambda_{k,l} = (2\pi a l/\beta)^2 + k^2,\ k = 1, ..., \infty,\ l = -\infty, ...\infty$$

If $p = 2q + 1,\ q \in \mathbb{N}$

$$d_k = \begin{cases} k\left[\frac{k}{2q+1}\right], & \text{for} \quad k - (2q+1)\left[\frac{k}{2q+1}\right] \quad \text{even} \\ k\left(\left[\frac{k}{2q+1}\right] + 1\right), & \text{for} \quad k - (2q+1)\left[\frac{k}{2q+1}\right] \quad \text{odd} \end{cases}$$

If $p = 2q,\ q \in \mathbb{N}$

$$d_k = \begin{cases} 0, & \text{for} \quad k \quad \text{even} \\ k\left(2\left[\frac{k}{2q}\right] + 1\right), & \text{for} \quad k \quad \text{odd} \end{cases}$$

Effective action $L(2q+1,1)\times S^1$

$$d_k = \begin{cases} k\left[\frac{k}{2q+1}\right], & \text{for} \quad k-(2q+1)\left[\frac{k}{2q+1}\right] \quad \text{even} \\ k\left(\left[\frac{k}{2q+1}\right]+1\right), & \text{for} \quad k-(2q+1)\left[\frac{k}{2q+1}\right] \quad \text{odd} \end{cases}$$

So, by writing $k = n(2q+1)+m$ with $n = 0, 1, ..., \infty$ and $m = 0, 1, ..., 2q$,

$$d_k = \begin{cases} k\left(\frac{k-m}{2q+1}\right), & \text{for} \quad m \quad \text{even} \\ k\left(\frac{k-m}{2q+1}+1\right), & \text{for} \quad m \quad \text{odd} \end{cases}$$

This last expression shows that the first term in both lines, when summed over all possible values of m, will reproduce the degeneracy of S^3, divided by the order of the cyclic group.

Low temperature

$$\delta\,\zeta_{S^3/Z_{2q+1}}(s) =$$

$$-\frac{(\mu\,a)^{2s}}{2q+1}\sum_{\substack{l=-\infty \\ n=0}}^{\infty}\left\{\sum_{m=1}^{q} 2m\,(n(2q+1)+2m)\left\{(n(2q+1)+2m)^2+\left(\frac{2la\pi}{\beta}\right)^2\right\}^{-s}\right.$$

$$\left.+\sum_{m=0}^{q-1} 2(m-q)\,(n(2q+1)+2m+1)\left\{(n(2q+1)+2m+1)^2+\left(\frac{2la\pi}{\beta}\right)^2\right\}^{-s}\right\}$$

Now, replacing m with $q-m$ in the second term and extracting a convenient overall factor, we have

$$\delta\,\zeta_{S^3/Z_{2q+1}}(s) =$$

$$-\frac{(\mu\,\beta)^{2s}}{(q+\frac{1}{2})(2\pi)^{2s}}\sum_{\substack{l=-\infty \\ n=0}}^{\infty}\left\{\sum_{m=1}^{q} m\,(n(2q+1)+2m)\left\{\left(\frac{\beta}{2a\pi}\right)^2(n(2q+1)+2m)^2+l^2\right\}^{-s}\right.$$

$$\left.-\sum_{m=1}^{q} m\,((n+1)(2q+1)-2m)\left\{\left(\frac{\beta}{2a\pi}\right)^2((n+1)(2q+1)-2m)^2+l^2\right\}^{-s}\right\}$$

The infinite sum over l can be inverted using property of Jacobi theta function,

$$\delta S_{\mathrm{eff},\,\mathrm{S}^3/\mathrm{Z}_{2q+1}} =$$

$$\frac{(2q+1)\beta}{a\pi^3} \sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-3} \sin\left(\frac{2\pi n m}{q+\frac{1}{2}}\right)$$

$$+ \quad 2\sum_{m=1}^{q} m \quad \sum_{\substack{l=1 \\ n=0}}^{\infty} \quad l^{-1} \left\{ \left[n + \frac{m}{q+\frac{1}{2}}\right] e^{-\frac{\beta l(2q+1)}{a}\left[n+\frac{m}{q+\frac{1}{2}}\right]} \right.$$

$$\left. - \quad \left[n + 1 - \frac{m}{q+\frac{1}{2}}\right] e^{-\frac{\beta l(2q+1)}{a}\left[n+1-\frac{m}{q+\frac{1}{2}}\right]} \right\}$$

High temperature

$$\zeta_{S^3/Z_{2q+1}}(s) = \frac{\zeta_{S^3}(s)}{2q+1} + \delta\,\zeta_{S^3/Z_{2q+1}}(s)$$

Similar treatment (now inverting series in k) gives

$$\delta S_{\mathrm{eff},\,\mathrm{S^3/Z_{2q+1}}} = \frac{1}{\pi}\sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-2}\,\sin\left(\frac{2\pi\,n\,m}{q+\frac{1}{2}}\right)$$

$$+\frac{1}{\pi(q+\frac{1}{2})}\sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-2} \sin\left(\frac{2\pi\,n\,m}{q+\frac{1}{2}}\right)\sum_{l=1}^{\infty}\left(1+\frac{2\pi^2\,n\,ma}{(q+\frac{1}{2})\beta}\right)e^{-\frac{2\pi^2 a l n}{(q+\frac{1}{2})\beta}}$$

Thermodynamic properties $L(2q+1,1)$

Zero temperature

$$F = E = \frac{1}{240(2q+1)a} + \frac{(2q+1)}{a\pi^3}\sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-3} \sin\left(\frac{2\pi n m}{q+\frac{1}{2}}\right) \qquad S = 0$$

Infinite temperature limit

$$F = \frac{\zeta_R(3)}{4\beta(2q+1)\pi^2} + \frac{1}{\pi\beta}\sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-2} \sin\left(\frac{2\pi\, n\, m}{q+\frac{1}{2}}\right) - \frac{\pi^4}{45(2q+1)}\frac{a^3}{\beta^4}$$

$$E = \frac{\pi^4 a^3}{15(2q+1)\beta^4}$$

$$S = -\frac{\zeta_R(3)}{4(2q+1)\pi^2} - \frac{1}{\pi}\sum_{m=1}^{q} m \sum_{n=1}^{\infty} n^{-2} \sin\left(\frac{2\pi\, n\, m}{q+\frac{1}{2}}\right) + \frac{4\pi^4}{45(2q+1)}\left(\frac{a}{\beta}\right)^3$$

Again, agreement with third law and topological entropy.

Some final comments

- Same treatment of $L(2q,1) = S^3/Z_{2p}$ gives

  Zero temperature

$$F = E = \frac{-7}{240aq} + \frac{2q}{a\pi^3}\sum_{m=1}^{q-1} m \sum_{n=1}^{\infty} n^{-3} \sin\left(\frac{2\pi n(m+\frac{1}{2})}{q}\right) \qquad S = 0$$

  Infinite temperature limit

$$\begin{aligned}
F &= -3\frac{\zeta_R(3)}{4q\beta\pi^2} - \frac{\pi^4 a^3}{90q\beta^4} + \frac{1}{\pi\beta}\sum_{m=1}^{q-1} m \sum_{n=1}^{\infty} n^{-2} \sin\left(\frac{2\pi\, n(m+\frac{1}{2})}{q}\right), \\
E &= \frac{\pi^4 a^3}{30q\beta^4}, \\
S &= \frac{3\zeta_R(3)}{4q\pi^2} + \frac{2\pi^4}{45q}\left(\frac{a}{\beta}\right)^3 - \frac{1}{\pi}\sum_{m=1}^{q-1} m \sum_{n=1}^{\infty} n^{-2} \sin\left(\frac{2\pi\, n(m+\frac{1}{2})}{q}\right)
\end{aligned}$$

  Note, again, topological entropy and agreement with third law

- Thermal Casimir effect for perfectly conducting planes:

  X. Zhai, Y. Yang and J. Lai, Int. Journ. Mod. Phys. 7, 202 (2012)

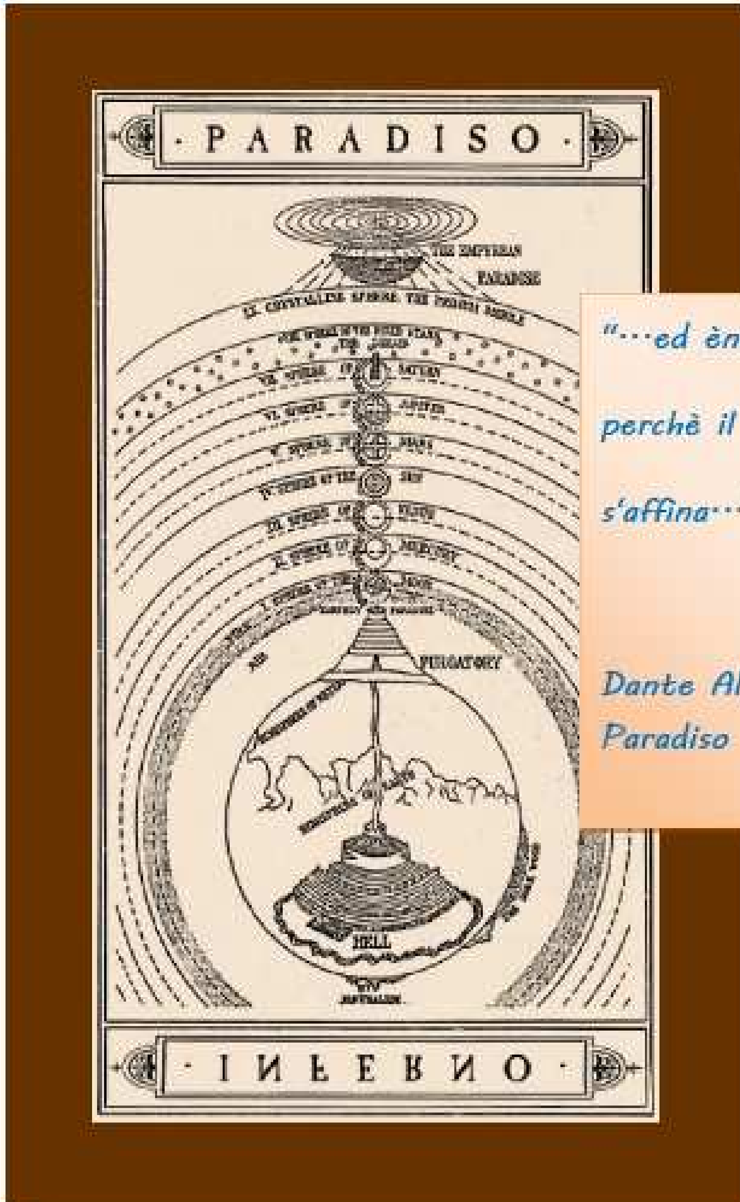
  Zero temperature

$$F/A = \frac{-\pi^2}{720a^3} - \frac{1}{4\beta^3}\sum_{n=1}^{\infty} n^{-3}\coth\left(\frac{n\pi\beta}{2a}\right) - \frac{\pi^3}{180a\beta^2}\sum_{n=1}^{\infty}\frac{n^{-2}}{\sinh^2\left(\frac{n\pi\beta}{2a}\right)}$$

  Infinite temperature limit

$$F/A = \frac{-\pi^2 a}{45\beta^4} - \frac{1}{8\pi a^2\beta}\sum_{n=1}^{\infty} n^{-3}\coth\left(\frac{2an\pi}{\beta}\right) - \frac{1}{4a\beta^2}\sum_{n=1}^{\infty}\frac{n^{-2}}{\sinh^2\left(\frac{2an\pi}{\beta}\right)}$$

  Inversion parameter $\delta = \frac{2a}{\beta}$ $a^3 F$ satisfies inversion rule

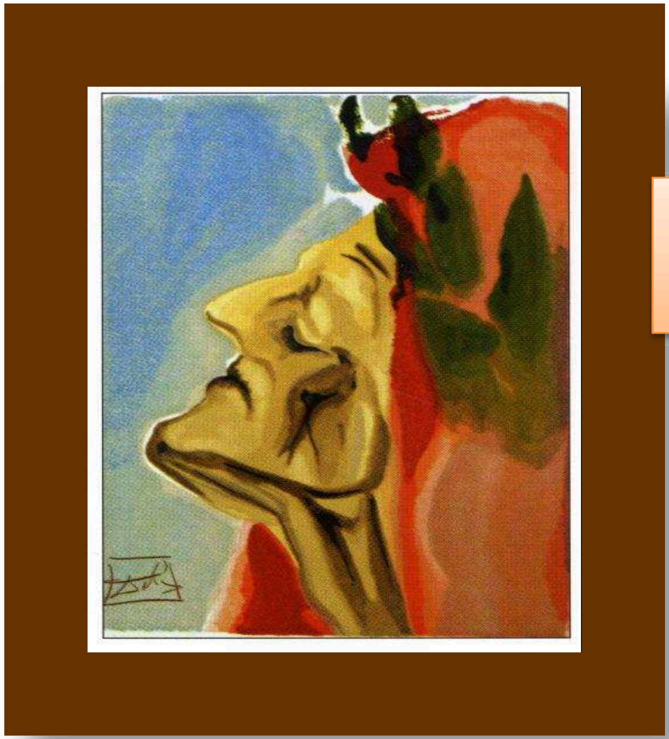


"…ed ènne dolce così fatto scemo

perchè il ben nostro in questo ben

s'affina…"

Dante Alighieri, Divina Commedia,
Paradiso – Canto XX



Salvador Dalí

Dante in Doubt, 1965

"...the incompleteness of our knowledge is a sweetness,

for our good is then refined..."

Dante Alighieri, Divina Commedia, Paradiso, Canto XX