Vacuum energy and Casimir forces: an introduction

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The ideal Casimir force

A universal effect from confinement of vacuum fluctuations:

\[ F_{\text{Cas}} = -\frac{dE_{\text{Cas}}}{dL}, \quad E_{\text{Cas}} = -\frac{\hbar c^2 A}{720L^3} \]

- Here written for
  - Perfectly parallel plane mirrors
  - Perfectly reflecting mirrors
  - Null temperature

- Attractive force (negative pressure)

\[ F_{\text{Cas}} = P_{\text{Cas}} A, \quad P_{\text{Cas}} = -\frac{\hbar c^2}{240L^4} \]

\[ |P_{\text{Cas}}| \sim 1\text{ mPa} \]


In order to obtain a finite result it is necessary to multiply the integrands by a function \( f(k/k_m) \) which is unity for \( k \ll k_m \) but tends to zero sufficiently rapidly for \( k/k_m \rightarrow \infty \), where \( k_m \) may be defined by \( f(1) = \frac{1}{2} \).

The physical meaning is obvious: for very short waves (X-rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate.

We are thus led to the following conclusions. There exists an attractive force between two metal plates which is independent of the material of the plates as long as the distance is so large that for wave, lengths comparable with that distance the penetration depth is small compared with the distance. This force may be interpreted as a zero point pressure of electromagnetic waves.


The Casimir force as an effect in Quantum Optics

- Many ways to calculate the Casimir effect
  - « Quantum Optics » approach used here
    - Field fluctuations pervade empty space
    - They exert radiation pressure on mirrors
    - Force = pressure difference between inner and outer sides of the mirrors
  - « Scattering approach »
    - Mirrors characterized by scattering amplitudes
    - Solves the high-frequency problem
    - Gives results for real mirrors

The Casimir force (real case)

- Real mirrors not perfectly reflecting
  - Force depends on non universal properties of the material plates used in the experiments
- Experiments performed at room temperature
  - Effect of thermal field fluctuations to be added to that of vacuum fluctuations
- Effects of geometry and surface physics
  - Plane-sphere geometry used in recent precise experiments
  - Surfaces not ideally plane: roughness …


- Theory has to be extended to non-specular scattering


The early history of quantum fluctuations

1900: Law for blackbody radiation energy per mode (Planck)

\[ E = \bar{n}_\omega \hbar \omega, \quad \bar{n}_\omega = \frac{1}{e^{\hbar \omega/k_B T} - 1} \]

1905: Derivation of this law from energy quanta (Einstein)

1912: Introduction of zero-point fluctuations (zpf) for matter (Planck)

\[ E = \bar{n}_\omega \hbar \omega + \frac{1}{2} \hbar \omega \]

1913: First correct demonstration of zpf (Einstein and Stern)

\[ \left( \bar{n}_\omega + \frac{1}{2} \right) \hbar \omega = k_B T + O \left( \frac{1}{T} \right), \quad T \to \infty \]

1914: Debye predicts effects of zpf on X-ray diffraction

1917: Quantum transitions between stationary states (Einstein)

1924: Quantum statistics for “bosons” (Bose and Einstein)

More history on vacuum fluctuations

1925-…: Quantum Mechanics confirms the existence of vacuum fluctuations (Heisenberg, Dirac and many others)

- Quantum electromagnetic field
  - Each mode = an harmonic oscillator

\[ E = E_1 \cos(\omega t) + E_2 \sin(\omega t) \]

\[ \Delta E_1 \Delta E_2 \geq \epsilon_\omega^2 \]

- Vacuum = ground state for all modes
- Fluctuation energy per mode \( \frac{1}{2} \hbar \omega \)

1945-…: Quantum Field Theory studies the effects of vacuum fluctuations in microphysics

1960-…: Quantum Optics studies thoroughly the properties and consequences of electromagnetic vacuum fluctuations

The puzzle of vacuum energy

1916: zp fluctuations for the electromagnetic fields lead to a BIG problem for vacuum energy (Nernst)

\[ e = \sum_{\text{modes}} \bar{n}_\omega \hbar \omega + \sum_{\text{modes}} \frac{\omega_{\text{max}} \hbar \omega}{2} = \frac{\pi^2 (k_B T)^4}{15 (hc)^3} + \frac{(\hbar \omega_{\text{max}})^4}{8\pi (hc)^3} \]

From conservative estimations of the energy density in vacuum…

Bound on vacuum energy density in solar system

Cutoff at the energy in accelerators (TeV)

\[ \frac{e_{\text{observ}}}{e_{\text{calcul}}} \sim 10^{-40} \]

…to the largest ever discrepancy between theory and experiment!

Now measured cosmic vacuum energy density

Cutoff at the Planck energy

\[ \frac{e_{\text{observ}}}{e_{\text{calcul}}} \sim 10^{-120} \]


The puzzle of vacuum energy

- Standard position for a large part of the 20th century
  [For the fields,] « it should be noted that it is more consistent, in contrast to the material oscillator, not to introduce a zero-point energy of $\frac{1}{2} h \nu$ per degree of freedom.
  For, on the one hand, the latter would give rise to an infinitely large energy per unit volume due to the infinite number of degrees of freedom, on the other hand, it would be in principle unobservable since nor can it be emitted, absorbed or scattered and hence, cannot be contained within walls and, as is evident from experience, neither does it produce any gravitational field. »

- Problem not yet solved, leads to many ideas, for example

When setting the cutoff to fit the cosmological observations (dark energy), one finds a length scale $\lambda = 85 \mu$m below which gravity could be affected.


A simple derivation of the Casimir effect

- Quantum field theory on 1d line
  - Counter-propagating scalar fields on the 1d line
  - Point-like "mirrors" couple the two propagation directions

- Calculations much easier than for the real case
  - Many concepts and methods are the same
  - Some results will be changed for the real 3d case (see the next lectures)

- More on the topic (many references in these papers)

Exclusion zone for Yukawa parameters


All available in the arXiv
Quantum field theory on the 1D line

- D’Alembert wave equation
  initially written for a string with 2 transverse vibrations
  written here for a single vibration (a single potential)

\[ \left[ \partial_t^2 - c^2 \partial_x^2 \right] \Phi(t, x) = 0 \]

- General solution given by the superposition of two counter-propagating waves
  \[ \Phi(t, x) = \varphi^+ \left( t - \frac{x}{c} \right) + \varphi^- \left( t + \frac{x}{c} \right) \]

- Canonical commutation relations \( \Leftrightarrow \) modes for this simplest version of quantum field theory


Normal modes

- Fourier decomposition of the fields
  \[ \varphi^n (u) - \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \varphi^n [\omega] e^{-i\omega u} \quad \varphi^n (t) \in \mathbb{R} \]

- Annihilation and creation operators correspond respectively to positive and negative frequencies
  \[ \varphi^n (u) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{\hbar}{2\omega}} \left( a_{\omega, \eta} e^{-i\omega u} + a_{\omega, \eta}^\dagger e^{i\omega u} \right) \]
  \[ \varphi^n [\omega] = \sqrt{\frac{\hbar}{2\omega}} \left( \theta(\omega) a_{\omega, \eta} + \theta(-\omega) a_{\omega, \eta}^\dagger \right) \]

- Canonical commutation relations
  \[ \left[ a_{\omega, \eta}, a_{\omega', \eta'}^\dagger \right] = 2\pi \delta (\omega - \omega') \delta_{\eta, \eta'} \]
  \[ \left[ a_{\omega, \eta}, a_{\omega', \eta'} \right] = \left[ a_{\omega, \eta}^\dagger, a_{\omega', \eta'}^\dagger \right] = 0 \]

Vacuum fluctuations

- Annihilation operators vanish in the vacuum state \( | \text{vac} \rangle \)
  \[ a_{\omega, \eta} | \text{vac} \rangle = 0 \]

- Creation operators do not
  \[ \left( a_{\omega, \eta}^\dagger a_{\omega', \eta'}^\dagger a_{\omega, \eta} | \text{vac} \rangle = 2\pi \delta (\omega - \omega') \delta_{\eta, \eta'} | \text{vac} \rangle \]

- All field correlations in vacuum \( \langle \ldots \rangle_{\text{vac}} = \langle \text{vac} | \ldots | \text{vac} \rangle \)

- Energy density for the wave equation
  \[ e(t, x) = \frac{1}{2} \left( \partial_t \Phi \right)^2 + \frac{1}{2} \left( c \partial_x \Phi \right)^2 \]

- General solution given by the superposition of counter-propagating energies
  \[ e(t, x) = e^+ (u_+) + e^- (u_-) \]
  \[ e^+ (u) = \left( \partial_u \varphi^+ (u) \right)^2 \]
  \[ e^- (u) = \left( \partial_u \varphi^- (u) \right)^2 \]

Spectral decomposition of energy density

- Fourier decomposition of the fields:
  \[ \varphi^\eta(u) = \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\frac{\hbar}{2\omega}} \left( a_{\omega,\eta} e^{-i\omega u} + a^\dagger_{\omega,\eta} e^{i\omega u} \right) \]
  \[ \partial_u \varphi^\eta(u) = \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\frac{\hbar}{2\omega}} \left( -i a_{\omega,\eta} e^{-i\omega u} + i a^\dagger_{\omega,\eta} e^{i\omega u} \right) \]

- Spectral decomposition of energies:
  \[ e^\eta(u) = (\partial_u \varphi^\eta(u))^2 \]
  \[ \langle e^\eta(u) \rangle_{\text{vac}} = \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty \frac{d\omega'}{2\pi} \sqrt{\frac{\hbar}{2\omega}} \sqrt{\frac{\hbar}{2\omega'}} \left\langle a_{\omega,\eta} a^\dagger_{\omega',\eta'} \right\rangle_{\text{vac}} \]
  \[ = \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar}{2\omega} \text{ Energy density per field mode} \]

One mirror on a 1D line

- One mirror on a 1D line
  \[ \begin{align*}
  \varphi^+_{\text{in}} & \quad \varphi^+-_{\text{out}} \\
  \varphi^-_{\text{in}} & \quad \varphi^--_{\text{out}}
  \end{align*} \]

- Properties of the S-matrix
  - For simplicity, we suppose the scattering matrix \( S_1 \) to have a symmetrical form:
    \[ r_1 \text{ and } t_1 \text{ defined for a mirror at } q=0 \]
    \[ S_1[\omega] = \begin{pmatrix} t_1[\omega] & r_1[\omega] e^{-2ikq} \\ r_1[\omega] e^{2ikq} & t_1[\omega] \end{pmatrix} \]
  - Properties:
    - amplitudes real in the time domain
    - scattering causal
    - scattering unitary
    - scattering vanishes at the high-frequency limit

A simple example

- Wave equation with an obstacle on the line
  \[ \left[ \partial_t^2 - c^2 \partial_x^2 + 2c \Omega \delta(x-q) \right] \Phi(t,x) = 0 \]

- Solution
  \[ r_1[\omega] = \frac{\Omega}{i\omega+\Omega} \]
  \[ t_1[\omega] = \frac{\omega}{i\omega-\Omega} \]
  Analogy: Quantum reflection over an attractive square well in the Dirac limit, with a very large depth and a very narrow width

- General properties obeyed, in particular
  - scattering causal \( r_1 \) has its pole at \( \omega = -i\Omega \), \( \Omega > 0 \)
  - scattering unitary \( r_1 r_1^* + t_1 t_1^* = 1 \), \( r_1 t_1^* + t_1 r_1^* = 0 \)
  - scattering vanishes at the high-frequency limit \( \lim_{\omega \to \infty} r_1[\omega] = 0 \), \( \lim_{\omega \to \infty} t_1[\omega] = 1 \)


Force on one mirror

- Force = difference of energy densities on left and right hand sides of the mirror
  \[ F' = \frac{e_L - e_R}{c} \]
  \[ e_L = e_{in}^+(u_+) + e_{out}^-(u_-) \]
  \[ e_R = e_{in}^- (u_-) + e_{out}^+ (u_+) \]

- For a perfect mirror, output density = input density on each side
  \[ e_{out}^-(t+\frac{q}{c}) = e_{in}^+(t-\frac{q}{c}) \]
  \[ e_{out}^+(t-\frac{q}{c}) = e_{in}^-(t+\frac{q}{c}) \]

Mean force on one mirror

- Mean energy densities on the two sides are infinite and equal
  \[ \left\langle e_L \right\rangle_{vac} = \left\langle e_R \right\rangle_{vac} = \int_0^{\infty} \frac{d\omega}{2\pi} h\omega \]

- Mean force vanishes on a mirror at rest in vacuum
  \[ \left\langle F \right\rangle_{vac} = \left\langle e_L - e_R \right\rangle_{vac} = 0 \]

- Homework (your choice!):
  - For non perfect mirrors, write the output energy densities in terms of the input ones and of coefficients of the S-matrix
  - Using unitarity of the S-matrix, prove that the mean force vanishes


Two mirrors on a line

- Two mirrors on a 1d line
- Each one couples rightward and leftward propagation
- Intra-cavity fields besides input and output fields
  \[ \Phi(t,x) = \theta(q_1-x) (\varphi_{in}^+(u_+) + \varphi_{out}^-(u_-)) \]
  \[ + \theta(x-q_1) \theta(q_2-x) (\varphi_{cav}^+(u_+) + \varphi_{cav}^-(u_-)) \]
  \[ + \theta(x-q_2) (\varphi_{out}^+(u_+) + \varphi_{in}^-(u_-)) \]

- The effect of the cavity is described by a global scattering matrix \( S \) and a resonance matrix \( R \)
  \[ \begin{pmatrix} \varphi_{out}^- \\ \varphi_{out}^+ \end{pmatrix} = S \begin{pmatrix} \varphi_{in}^- \\ \varphi_{in}^+ \end{pmatrix} \quad \begin{pmatrix} \varphi_{cav}^- \\ \varphi_{cav}^+ \end{pmatrix} = R \begin{pmatrix} \varphi_{in}^- \\ \varphi_{in}^+ \end{pmatrix} \]
**Global S-matrix**

- The global S-matrix can be evaluated from the elementary matrices $S_1$ and $S_2$.
- Result (all amplitudes depend on frequency)

$$S = \frac{1}{d} \left( \begin{array}{ccc}
   t_1 & t_2 & e^{-ikL} + t_2 r_1 e^{ikL} \\
   d r_1 e^{-ikL} + t_2 r_2 e^{ikL} & t_1 t_2 & 0 \\
   t_2 r_1 e^{ikL} & 0 & t_2
\end{array} \right)$$

$$d[\omega] = 1 - r[\omega] e^{2ikL}, \quad r[\omega] \equiv r_1[\omega] r_2[\omega]$$

- The matrix $S$ is unitary, like $S_1$ and $S_2$

$$S[\omega] S[\omega]^\dagger = I$$

**Energy densities on the outer sides**

- We calculate the energy densities on the outer sides of the mirrors 1 and 2 (left and right sides of the cavity)

$$e_L = e_{\text{in}}^+ (u_+) + e_{\text{out}}^- (u_-)$$

$$e_R = e_{\text{in}}^- (u_-) + e_{\text{out}}^+ (u_+)$$

- Same result as for a single mirror

$$\langle e_L \rangle_{\text{vac}} = \langle e_R \rangle_{\text{vac}} = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega$$

- Again the consequence of unitarity of the $S$ matrix

**Resonance matrix**

- The resonance matrix can be evaluated from the elementary matrices $S_1$ and $S_2$.
- Result (all amplitudes depend on frequency)

$$R = \frac{1}{d} \left( \begin{array}{ccc}
   t_1 & t_2 & e^{2ikL} \\
   t_2 r_2 e^{ikL} & t_2 & 0 \\
   t_2 r_1 e^{-ikL} & 0 & t_2
\end{array} \right)$$

$$d[\omega] = 1 - r[\omega] e^{2ikL}, \quad r[\omega] \equiv r_1[\omega] r_2[\omega]$$

- Result written here for $q_2 = \frac{L}{2} = - q_1$

**Energy densities on the inner sides**

- We calculate the energy densities on the inner sides of the mirrors 1 and 2 (intra-cavity sides)

$$e_{\text{cav}} = e_{\text{in}}^+ + e_{\text{out}}^-$$

- Result

$$\langle e_{\text{cav}} \rangle_{\text{vac}} = \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega g[\omega]$$

$$g[\omega] = \frac{|R_{11}|^2 + |R_{12}|^2 + |R_{21}|^2 + |R_{22}|^2}{2} = \frac{1 - |r|^2}{|1 - r e^{2ikL}|^2}$$

- Modification of the energy density inside the cavity with respect to input fields

$$r[\omega] \equiv r_1[\omega] r_2[\omega]$$

- Equivalent to a change of the density of states
Casimir force as a radiation pressure

- The force on each mirror is the difference of radiation pressures on the inner and outer sides:
  \[
  F_1 = \frac{e_L - e_{cav}}{c}, \quad F_2 = \frac{e_{cav} - e_R}{c}
  \]
  \[
  F = \langle F_2 \rangle_{\text{vac}} - \langle F_1 \rangle_{\text{vac}} = \int_0^\infty \frac{d\omega}{2\pi c} \hbar \omega (g[\omega] - 1)
  \]
- Repulsive contributions for resonant frequencies
- Attractive contributions for frequencies out of resonance
- The net force is the sum of contributions of all modes

Casimir force and causality

- The force may be written in terms of a causal “loop function”:
  \[
  g[\omega] = 1 + f[\omega] + (f[\omega])^* = \frac{r[\omega]e^{2ikL}}{1 - r[\omega]e^{2ikL}}
  \]
  \[
  g = \frac{1 - |r|^2}{|1 - re^{2ikL}|^2}
  \]
- So that
  \[
  F = \int_0^\infty \frac{d\omega}{2\pi c} \hbar \omega (g[\omega] - 1) = I_r + I_r^*
  \]
  \[
  I_r = \int_0^\infty \frac{d\omega}{2\pi c} \hbar \omega f[\omega]
  \]
- Using the causality properties of the scattering amplitudes, the last integral can be written over imaginary frequencies (Wick rotation)

Wick rotation

- The loop function is analytic in the upper half of the complex plane:
  \[
  \text{Im} \omega \geq 0 \quad \omega = i\xi
  \]
- Cauchy theorem:
  \[
  I_r + I_{\infty} + I_i = 0
  \]
- High frequency transparency:
  \[
  I_{\infty} = 0
  \]
- The integral over real frequencies can be translated into an integral over imaginary frequencies:
  \[
  I_r = -I_i = -\int_0^{\infty} \frac{d\xi}{2\pi c} \hbar \xi f[i\xi]
  \]
  \[
  F = 2I_r = -\int_0^{\infty} \frac{d\xi}{\pi c} \hbar \xi f[i\xi]
  \]

Final expression (1d)

- Casimir force between 2 mirrors on the 1d line:
  \[
  F = -\frac{\hbar}{\pi c} \int_0^{\infty} \frac{d\xi}{1 - r[i\xi] e^{-2\kappa L}} = -\frac{\hbar c \pi}{24L^2} \int_0^{\infty} \frac{x \, dx}{e^x - 1} = \frac{\pi^2}{6}
  \]
  \[
  \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
  \]
- For “real” mirrors (r<1):
  - the integral is always regular
  - it has a smaller magnitude than for perfect mirrors

Outline of the three lectures

✓ Introduction
  ✓ An introduction to the Casimir effect
  ✓ A short history of field fluctuations in electromagnetic vacuum

✓ A simple derivation of the Casimir effect
  ✓ The simplest quantum field theory: scalar fields on the 1d line
  ✓ Introduction to the scattering approach

✓ The Casimir pressure in 3d space
  ✓ Vacuum and thermal fluctuations of electromagnetic fields
  ✓ Models and results for "real" mirrors

  ▪ Beyond the plane-plane geometry
    ▪ General (non specular) scattering formula
    ▪ Explicit calculations beyond the proximity approximation

The Casimir force in 3d space

✓ Quantum field theory of the electromagnetic field
  ▪ Vacuum and thermal fluctuations of electromagnetic fields
  ▪ Energy density and radiation pressure in 3d space
  ▪ Casimir pressure between 2 parallel plane mirrors

✓ Models and results for real mirrors
  ▪ Models for metallic mirrors
  ▪ Thermal effects and dissipation
  ▪ High-temperature limit

✓ More on the topic (many references in these papers)

Free electromagnetic field in 3D space

✓ Free modes for Maxwell equations
  ▪ Propagation at the speed of light
  ▪ with arbitrary frequency and direction, or arbitrary wave-vector
  \[ \omega^2 = c^2 \left( k_x^2 + k_y^2 + k_z^2 \right) \]

  \[ \hat{k} = \frac{c}{\omega} \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} \]

✓ Unit wave-vector, unit TE and TM polarization vectors form an orthonormal basis for each mode

\[ \hat{\alpha}^{TE} = \hat{\beta}^{TM} = \begin{pmatrix} -\sin \varphi \\ -\cos \varphi \\ 0 \end{pmatrix}, \quad \hat{\alpha}^{TM} = -\hat{\beta}^{TE} = \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix} \]

Free electric and magnetic fields

✓ Mode labelling

\[ m \equiv (k, k_z, p) \quad \text{or} \quad m \equiv (k, \omega, \eta, p) \]

\[ k = k_x \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad k_z = \eta \sqrt{\frac{\omega^2 - k_x^2}{c^2}} \]

\[ \sum_m A \sum_p \int \frac{d^2k}{4\pi^2} \int \frac{dk_z}{2\pi} = A \sum_p \int \frac{d^2k}{4\pi^2} \sum_m \int \frac{dk_z}{2\pi} \]

✓ Electric and magnetic fields

\[ E = \sum_m \sqrt{\frac{\hbar \omega}{2\varepsilon_0}} \hat{\alpha}_m \left( -ia_m e^{-i\omega t + ikx + ikz} + ia^+_m e^{i\omega t - ikx - ikz} \right) \]

\[ B = \sum_m \sqrt{\frac{\hbar \omega \mu_0}{2}} \hat{\beta}_m \left( -ia_m e^{-i\omega t + ikx + ikz} + ia^+_m e^{i\omega t - ikx - ikz} \right) \]

\[ \hat{\beta}_m = \hat{k}_m \times \hat{\alpha}_m \]
Energy density

- Energy density for the Maxwell equations
  - Given by the « 00 » component of the Maxwell stress tensor
    \[ e = \frac{\varepsilon_0 E \cdot E}{2} + \frac{B \cdot B}{2\mu_0} \]
  - Result of the evaluation in vacuum
    \[ \langle e \rangle_{\text{vac}} = \sum_m \sum_{m'} \sqrt{\frac{\hbar \omega}{2}} \sqrt{\frac{\hbar \omega'}{2}} \langle a_m a_{m'}^\dagger \rangle_{\text{vac}} = \sum_m \frac{\hbar \omega_m}{2} \]
  - Result of the evaluation at thermal equilibrium
    \[ \langle e \rangle_T = \sum_m \hbar \omega_m \left( \frac{1}{2} + n_m \right) \]
    \[ n_m = \frac{1}{e^{\hbar \omega / k_B T} - 1} \]

Radiation pressure

- Radiation pressure on a surface of constant z
  - Given by the « zz » component of the Maxwell stress tensor
    \[ \frac{\varepsilon_0}{2} \left( E_x E_x + E_y E_y - E_z E_z \right) + \frac{B_x B_x + B_y B_y - B_z B_z}{2\mu_0} \]
  - Result of the evaluation in vacuum
    \[ \frac{1}{A} \sum_m \frac{\hbar \omega_m}{2} \cos^2 \theta_m \]
  - Result of the evaluation at thermal equilibrium
    \[ \frac{1}{A} \sum_m \hbar \omega_m \left( \frac{1}{2} + n_m \right) \cos^2 \theta_m \]

Scattering amplitudes for plane mirrors

- We consider plane mirrors (surfaces of constant z)
  - Invariance under lateral translations
  - Transverse wave-vector and polarization preserved
  - Specular scattering amplitudes which depend on frequency \( \omega \), polarization \( p \), incidence angle \( \theta \)
  - Only parameter affected is \( \eta \)

- Scattering matrix
  \[ \begin{pmatrix} E_{\text{out}}^+ \\ E_{\text{out}}^- \end{pmatrix} = S_1 \begin{pmatrix} E_{\text{in}}^+ \\ E_{\text{in}}^- \end{pmatrix} \]

Scattering on a cavity

- Calculations are the same as for the 1d case
  - Effect of the cavity described by a global scattering matrix and a resonance matrix
  \[ \begin{pmatrix} E_{\text{out}}^+ \\ E_{\text{out}}^- \end{pmatrix} = S \begin{pmatrix} E_{\text{in}}^+ \\ E_{\text{in}}^- \end{pmatrix} \]
  \[ \begin{pmatrix} E_{\text{cav}}^+ \\ E_{\text{cav}}^- \end{pmatrix} = R \begin{pmatrix} E_{\text{in}}^+ \\ E_{\text{in}}^- \end{pmatrix} \]
  - These matrices \( S \) and \( R \) can be evaluated from the elementary \( S \)-matrices associated with the mirrors 1 and 2


\( S \) and \( R \)-matrices

- Results
  - Scattering amplitudes for the mirrors 1 and 2 defined as if the mirrors were at \( z=0 \)
  - All amplitudes depend on frequency, polarization and incidence angle
  - Propagation phase-shifts determined by \( k_z = \pm |k_z| \)

\[
S = \frac{1}{d} \left( \begin{array}{c}
   t_1 t_2 e^{-i|k_z|L} + t_2^2 r_1 e^{i|k_z|L} \\
   t_1^2 r_2 e^{i|k_z|L}
\end{array} \right)
\]

\[
R = \frac{1}{d} \left( \begin{array}{c}
   t_1 \quad t_2^2 r_1 e^{i|k_z|L} \\
   t_1^2 r_2 e^{i|k_z|L}
\end{array} \right)
\]

\[
d = 1 - re^{2i|k_z|L} , \quad r \equiv r_1 r_2
\]

Radiation pressures

- Most calculations are the same as for the 1d case
  - Radiation pressures on the outer sides of the mirrors not affected
  - Radiation pressures on the inner sides of the mirrors modified
  - Change of the density of states inside the cavity

\[
P = \frac{1}{A} \sum_m \frac{\hbar \omega_m}{2} \cos^2 \theta_m \left( g_m - 1 \right)
\]

\[
g_m = \frac{1 - |r_m|^2}{|1 - r_m e^{2i|k_z|L}|^2} = 1 + f_m + f_m^* , \quad f_m = \frac{r_m e^{2i|k_z|L}}{1 - r_m e^{2i|k_z|L}}
\]

- So that

\[
\mathcal{P} = \mathcal{I}_r + \mathcal{I}_r^*
\]

\[
\mathcal{I}_r = \sum_p \int \frac{d^2k}{4\pi^2} \int \frac{d|k_z|}{2\pi} \frac{\hbar \omega_m}{\omega_m^2} \frac{c^2 k_z^2}{f_m}
\]

\[
= \sum_p \int \frac{d^2k}{4\pi^2} \int \frac{d\omega}{2\pi} \hbar |k_z| f_m
\]

- \( c^2 |k_z| \rightarrow \omega d\omega \)

Derivation of the Casimir pressure

- Some elements to be treated with greater care
  - Effect of dissipation and associated fluctuations
  - Contribution of evanescent modes

- Fluctuations-dissipation theorem
  - Vacuum fluctuations of the fields and fluctuations entering through the losses in matter
  - Expression of the density of states keeps the same expression in presence of dissipation

Evanescent modes

- Up to now, we have discussed ordinary waves
  - Real wave-vector \( k_z \) real \( \rightarrow \omega \geq c|k| \)

\[
|k| \equiv \sqrt{k_x^2 + k_y^2}
\]

- We now consider the contributions of evanescent waves
  - Imaginary wave-vector along \( z \) \( k_z \) imaginary \( \rightarrow 0 \leq \omega \leq c|k| \)

\[
\omega^2 = c^2 (k^2 + k_z^2)
\]

\[
\mathcal{I}_r = \sum_p \int \frac{d^2k}{4\pi^2} \int_0^\infty \frac{d\omega}{2\pi} \hbar |k_z| \frac{r^p_\omega e^{2i|k_z|L}}{1 - r^p_\omega e^{2i|k_z|L}}
\]

\[
\int_0^\infty + \int_{|k|} \text{ includes evanescent and ordinary waves}
\]

The loop function is analytic in the upper half of the complex plane
\[ \text{Im} \omega \geq 0 \]
- Cauchy theorem
  \[ J_r + J_\infty + J_i = 0 \]
- High frequency transparency
  \[ J_\infty = 0 \]
- The integral over real frequencies can be translated into an integral over imaginary frequencies
\[ P = 2J_r = -2\sum_p \int \frac{d^2k}{4\pi^2} \int_0^\infty \frac{d\xi}{2\pi} \frac{\mu_k}{2} \frac{r^p_k[i\xi]}{1 - r^p_k[i\xi]} e^{-2kL} \]
  \[ \kappa \text{ obtained from } k_z \text{ after the Wick rotation} \]
\[ \kappa = \sqrt{k^2 + \frac{\xi^2}{c^2}} \]

Models for reflection amplitudes
- Common model
  - bulk mirror (very thick slab)
  - local (reduced) dielectric response function \( \varepsilon[\omega] \)
  - reflection amplitudes on each mirror given by Fresnel laws
  \[ r^\text{TE}_r[\omega] = \frac{k_z - K_z}{k_z + K_z} \], \[ r^\text{TM}_r[\omega] = \frac{K_z - \varepsilon k_z}{K_z + \varepsilon k_z} \]
  \[ K_z = \sqrt{\frac{\omega^2}{c^2} - k^2} \text{, } k_z = \sqrt{\frac{\omega^2}{c^2} - k^2} \]
\[ r^\text{TE}_r[i\xi] = \frac{\kappa - K}{\kappa + K} \], \[ r^\text{TM}_r[i\xi] = \frac{K - \varepsilon[i\xi]}{K + \varepsilon[i\xi]} \]
  \[ K = \sqrt{\frac{\xi^2}{c^2} - k^2} \text{, } \kappa = \sqrt{\frac{\xi^2}{c^2} + k^2} \]
  \[ \Rightarrow \text{Dzyaloshinskii-Lifshitz-Pitaevskii formula} \]

Models for metallic mirrors
- Simple models for the (reduced) dielectric function for metals
  \( \varepsilon[i\xi] = \varepsilon[i\xi] + \sigma[i\xi] \)
  \[ \sigma[i\xi] = \frac{\omega^2_p}{\xi + \gamma} \]
  \[ \varepsilon[0] = \frac{\omega^2_p}{\varepsilon_0 m^*} \]
  \[ \sigma[0] = \frac{\omega^2_p}{\gamma} \]
- Drude parameters related to the density of conduction electrons and to the static conductivity
  \( \Rightarrow \) non null \( \gamma \)

Pressure between metallic mirrors \((T=0)\)

- Simplest model (no inter-band, no losses)
  - Lossless plasma model
    \[ \varepsilon[i\xi] = 1 + \frac{\omega_p^2}{\xi^2} \]
  - Plasma frequency and wavelength
    \[ \lambda_p = \frac{2\pi c}{\omega_p} \]
- Pressure reduction \(wrt\) ideal Casimir formula
  \[ \eta_p = \frac{P}{P_{\text{Cas}}} \]


The plasmon limit (short distances)

- For each metallic mirror, there are surface plasmons living on the interface with vacuum
- Plasmons on the two bulks are coupled by Coulomb law
  \[ \omega_{\pm} = \omega_0 \sqrt{1 \mp e^{-kL}} \]
- At short distances, the Casimir force can be seen as the effect of van der Waals coupling of surface plasmons
  \[ \varepsilon = A \int \frac{d^2 k}{(2\pi)^2} \left( \frac{\hbar \omega_+}{2} + \frac{\hbar \omega_-}{2} - 2\hbar \omega_0 \right) = -\frac{\hbar c \alpha^2 A}{480\lambda_p L^2} \]
  \[ \alpha = \frac{15\sqrt{2}}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{(4n-3)!!}{(4n-2)!!} \approx 1.193 \]
- The picture can be extended to arbitrary distances by taking into account plasmonic as well as photonic modes


Pressure between gold-covered mirrors \((T=0)\)

- More precise model for gold-covered mirrors
  - Integration of optical data
  - Extrapolation at low frequencies with a Drude model
  - Causality relations
    \[ \varepsilon[\omega] = \varepsilon_r[\omega] + i\varepsilon_i[\omega] \]
    \[ \rightarrow \varepsilon[i\xi] \]


Global behaviour well reproduced by the plasma model (at \(T=0\))

Integration of optical data necessary for precise predictions

More to come with the effect of thermal fluctuations

Gold mirrors (optical data)

Plasma model \(\lambda_p = 136\text{nm}\)

Casimir force at finite temperature (1d)

- The Casimir force at finite temperature is the sum over all field modes of the radiation pressure difference
- Written here for the 1d case

\[ F = \int_{0}^{\infty} \frac{d\omega}{2\pi c} \Pi[\omega] (g[\omega] - 1) \]

Field fluctuation energy in the two counter-propagating modes at frequency \( \omega \)
\[ \Pi[\omega] = 2\hbar \omega \left( \frac{1}{2} + \tanh \frac{\hbar \omega}{2k_B T} \right) \]

Cavity confinement effect


Cavity confinement effect

High-temperature limit (1d)

- At the high-temperature limit, the Matsubara poles with \( n > 0 \) give a negligible contribution
- Casimir force dominated by the term \( n = 0 \)
- The latter vanishes (1d) because \( \kappa_0 = 0 \)

\[ \lim_{T \to \infty} F = 0 \]

- In the high-temperature limit (= classical limit), the Casimir force vanishes
- Beware, this result is valid only in 1d!

Casimir force at \( T \neq 0 \) (1d)

- The Casimir force can be rewritten by using causality properties and transparency at high frequencies
- Important change: the function \( \Pi[\omega] \) has poles at regularly spaced imaginary frequencies (Matsubara frequencies)

\[ \omega_n = i\xi_n, \quad \xi_n = n \frac{2\pi k_B T}{\hbar} \]

After Wick rotation, the force is now a discrete sum over Matsubara frequencies

\[ F = -k_B T \sum_{n} \frac{2\kappa_n r[i\xi_n]}{1 - r[i\xi_n]} e^{-2\kappa_n L} \]

The limit \( T=0 \) gives the already known integral

\[ F_0 = -\frac{\hbar c}{\pi} \int_{0}^{\infty} d\xi \frac{\kappa r(i\xi)e^{-2\kappa L}}{1 - r(i\xi)e^{-2\kappa L}} \]

- With Fresnel reflection amplitudes for a bulk mirror, this is the Dzyaloshinskii-Lifshitz-Pitaevskii formula

Casimir pressure at \( T \neq 0 \) (3d)

- Same reasoning as previously
- \( \Pi[\omega] \) has poles at Matsubara frequencies

\[ \omega_n = i\xi_n, \quad \xi_n = n \frac{2\pi k_B T}{\hbar} \]

After Wick rotation, the pressure is a sum over Matsubara frequencies

\[ P = -k_B T \sum_{p} \int \frac{d^2k}{(2\pi)^2} \sum_{n} \frac{2\kappa_n r_p[k,i\xi_n]}{1 - r_p[k,i\xi_n]} e^{-2\kappa_n L} \]

The limit \( T=0 \) gives the known integral

\[ P_0 = -\frac{\hbar c}{\pi} \sum_{p} \int \frac{d^2k}{(2\pi)^2} \int d\xi \frac{\kappa r_p[k,i\xi]e^{-2\kappa L}}{1 - r_p[k,i\xi]e^{-2\kappa L}} \]

- With Fresnel reflection amplitudes for a bulk mirror, this is the Dzyaloshinskii-Lifshitz-Pitaevskii formula
Dzyaloshinskii-Lifshitz-Pitaevskii formula

- Pressure between plane dielectric bulks at T≠0
- Force per unit area on bodies 1 and 2 separated by a medium 3
- Media described by dielectric constants depending on frequency $\epsilon_1$

$$\begin{align*}
s_1 &= \sqrt{\frac{\epsilon_1}{\epsilon_3} - 1 + p^2} \\
s_2 &= \sqrt{\frac{\epsilon_2}{\epsilon_3} - 1 + p^2} \\
\xi_n &= \frac{2\pi nk_B T}{\hbar} \\
\epsilon_1 &= 1, 2, 3, \Xi = \epsilon_1(i\xi_n)
\end{align*}$$

$$F(l) = \frac{kT}{\pi\epsilon_3^3} \sum_{n=0}^\infty \epsilon_3^{2\xi_n^3} \int_1^\infty p^2 \left[ \left( \frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)} \right) \exp \left( \frac{2p\xi_n}{c} \sqrt{\epsilon_3} \right) - 1 \right]^{-1} \left( \frac{(s_1 - p)(s_2 - p)}{(s_1 - p\epsilon_1)(s_2 - p\epsilon_3)} \left( \frac{2p\xi_n}{c} \sqrt{\epsilon_3} \right) - 1 \right]^{-1} \, dp$$

(4.13)


Pressure between metallic mirrors at T≠0

- Explicit calculation for simple models of metallic mirrors
  - bound electrons disregarded here
  - Drude model for conductivity
    - lossless plasma model $\gamma = 0$
    - dissipative Drude model $\gamma \neq 0$

$$\begin{align*}
\varepsilon[i\xi] &= 1 + \frac{\sigma[i\xi]}{\xi} \\
\sigma[i\xi] &= \frac{\omega_p^2}{\xi + \gamma}
\end{align*}$$

- Strong correlation between the effects of thermal fluctuations and dissipation
  - factor 2 between the two estimations in spite of the very small relative value of the Drude dissipation parameter $\frac{\gamma}{\omega_p} \sim 0.004$ for gold

### High-temperature limit

- At the large-distance limit (or high-temperature limit) $2\pi k_B T L \gg \hbar c$

$$P = -k_B T \sum_p \int_0^\infty \frac{dk}{2\pi} \frac{k^2 r_k^p[0] e^{-2kL}}{1 - r_k^p[0] e^{-2kL}}$$

- With the lossless plasma model (as with perfect mirrors) $r_k^{TE}[0] = r_k^{TM}[0] = 1$
- With the dissipative Drude model $r_k^{TE}[0] = 0, r_k^{TM}[0] = 1$
- So that

$$\begin{align*}
\lim_{T \to \infty} P_{\gamma 
eq 0} &= -\frac{k_B T}{2\pi} \int_0^\infty \frac{k^2 \, dk}{e^{2kL} - 1} - \frac{k_B T}{8\pi L^3} \\
\lim_{T \to \infty} P_{\gamma = 0} &= 2 \lim_{T \to \infty} P_{\gamma \neq 0}
\end{align*}$$

\[\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202\]
Discussion

- DLP formula OK in the limits of its assumptions
  - bulk mirrors; local linear dielectric response
- The scattering formula allows one to accommodate more general cases for the reflection amplitudes
  - finite thickness, multilayer structure
  - non isotropic response, chiral materials
  - non local dielectric response
  - microscopic models of optical response …
- It can be generalized to more general geometries
  - non specular reflection for the mirrors (next lecture)


An example of layered mirrors: Si slabs

- Casimir force between Silicon slabs versus thickness

Outline of the three lectures

- Introduction
  - An introduction to the Casimir effect
  - A short history of field fluctuations in electromagnetic vacuum
- A simple derivation of the Casimir effect
  - The simplest quantum field theory: scalar fields on the 1d line
  - Introduction to the scattering approach
- The Casimir pressure in 3d space
  - Vacuum and thermal fluctuations of electromagnetic fields
  - Models and results for “real” mirrors
- Beyond the plane-plane geometry
  - General (non specular) scattering formula
  - Explicit calculations beyond the proximity approximation

Beyond the plane-plane geometry

- The general scattering formula
  - Phase-shift representation of the Casimir free energy
  - General formula for non specular scattering
- Explicit calculations beyond the proximity approximation
  - Rough and corrugated plates
  - Plane-sphere geometry
  - Applications
- More on the topic (many references in these papers)
We put the scattering system in a large box with periodic conditions. We calculate the modes in the large box, which are determined by the eigen-values of the S-matrix associated with the scatterer:

$$e^{i(kz + \delta_m)} = 1, \quad e^{i\delta_m} \text{ eigenvalue of } S$$

The change of free energy due to the presence of the scatterer is given by a sum over modes:

$$\Delta F \equiv \sum_m \hbar N_m \Delta \omega_m, \quad N_m \equiv \frac{1}{2} + \bar{n}_{\omega_m}$$

with the change of frequency determined by the phase-shifts $\delta_m$.

The phase-shift formula for free energy:

Using the expression written above for the S-matrix in terms of $S_1$ and $S_2$, we see that their determinants obey the relation:

$$\det S = (\det S_1) (\det S_2) \frac{(1 - re^{2ikzL})^*}{1 - re^{2ikzL}}$$

It follows that:

$$\Delta F_{12} = \Delta F_1 + \Delta F_2 + \mathcal{F}$$

with the interaction term:

$$\mathcal{F} = \sum_i \hbar N_m \ln \left( \frac{1 - re^{2ikzL})^*}{1 - re^{2ikzL}} \right)$$

The interaction free energy is obtained as a difference between different configurations:

$$\mathcal{F} = \Delta F_{12} - \Delta F_1 - \Delta F_2$$

$$= F_{12} - F_1 - F_2 + F_0$$

The phase-shift formula after Wick rotation:

After Wick rotation:

$$\mathcal{F} = \frac{k_B T}{A} \sum_n \text{Tr} \ln D[i\xi_n]$$

$$d\mathcal{F}[i\xi_n] = 1 - r^2[i\xi_n]e^{-2\kappa_nL}$$

Equivalent to the already written expression for the pressure:

$$P = \frac{\mathcal{F}}{A} = -\frac{\partial \mathcal{F}(L,T)}{\partial L} = -\frac{k_B T}{A} \sum_n 2\kappa_n r[i\xi_n]e^{-2\kappa_nL}$$

One may also define an entropy:

$$S = -\frac{\partial \mathcal{F}(L,T)}{\partial T}$$

and an “internal energy”:

$$\mathcal{E} = \mathcal{F} + TS = \mathcal{F} - T\frac{\partial \mathcal{F}(L,T)}{\partial T}$$

The general scattering formula:

We now consider the most general configuration with two scatterers at rest in electromagnetic field:

- free energy can still be written as a sum over phase-shifts
- phase-shifts correspond to eigen-values of a large S-matrix which describes non specular couplings between modes with different wave-vectors and polarizations (same frequency)
- causality and high-frequency transparency allow one to write the free energy as an integral over imaginary frequency:

$$\mathcal{F} = k_B T \sum_n \text{Tr} \ln D[i\xi_n]$$

$$d\mathcal{F}[i\xi_n] = e^{-\kappa L}$$

free propagators
diagonal on plane waves

$R_1, R_2$ reflection matrices
$D, R_1, R_2$ non diagonal matrices

Diffraction by rough or corrugated plates

- Transverse wave-vector is changed by diffraction on a non flat surface
  \[ \Delta k = \frac{2\pi}{\lambda_C} \]
  \( \lambda_C \) roughness or corrugation wave-vector
- This diffraction effect is not properly treated by the "proximity force approximation"
- We have calculated analytically the correction to Casimir force between rough plates from the general scattering formula, in a perturbative expansion in the roughness amplitudes
- Results available for perfect mirrors and for metallic mirrors

P.A. Maia Neto, A. Lambrecht, S. Reynaud, EPL 69 (2005) 924

Lateral Casimir force

- Symmetry broken for transverse translations \( \rightarrow \) Lateral forces
- After explicit calculations, the variation of energy with the corrugations is described by a spectral sensitivity
  \[ \delta E_{\text{corrug}} = a_1 a_2 \cos(k_C b) G_C(k_C) \]
- PFA is recovered at the limit of smooth corrugations \( k_C \rightarrow 0 \)
- Deviation from PFA is measured by the factor
  \[ \rho_C(k_C) \equiv \frac{G_C(k_C)}{G_C(0)} \]


Perturbation theory for corrugated plates

- Energy written here at zero temperature
  \[ \mathcal{E} = -\hbar \int_0^\infty \frac{d\xi}{2\pi} \text{Tr} \ln \mathcal{D}[i\xi] \quad , \quad \mathcal{D} = 1 - \mathcal{R}_1 e^{-\kappa L} \mathcal{R}_2 e^{-\kappa L} \]
- Small deformations wrt the plane-plane geometry
  - Corrugations on the two plates with small amplitudes
  - Study of the lateral force which appears when the corrugations are shifted wrt each other
  - Evaluation up to second order in the corrugation amplitudes in terms of the variations of the reflection matrices
  \[ \delta \mathcal{E}_{\text{corrug}} = -\hbar \int_0^\infty \frac{d\xi}{2\pi} \text{Tr} \left( \delta \mathcal{R}_1 e^{-\kappa L} \mathcal{D} \delta \mathcal{R}_2 e^{-\kappa L} \mathcal{D} \right) \]


Deviation from PFA

- Evaluation for a simple microscopic model
  - bulk metallic plates described by the plasma model
  - non specular reflection amplitudes calculated in the Rayleigh approximation
- Non trivial effects of geometry predicted
  \[ \rho_C(k_C) \equiv \frac{G_C(k_C)}{G_C(0)} < 1 \]

Vacuum-induced torques

- Symmetry broken for rotations → vacuum fluctuations induce a torque which tends to align two plates whose corrugations would not be parallel
- Casimir torque: transfer of angular momentum through the coupling to quantum vacuum fluctuations
- Plot:
  - Solid line: full theory (Gold)
  - Dotted line: PFA (Gold)
  - Dashed line: perfect mirrors

\[
\lambda_P = 136\text{nm}
\]
\[
L = 1\mu\text{m}
\]


Deep corrugations

- An exact expression can be obtained also for the force between nanostructured surfaces made of real materials with arbitrary corrugation depth \(a\), corrugation width \(d\) and distance \(L\)
- Explains the deviation from PFA seen in experiments
  

Comparison with experiments

- Force gradient between a sphere and a corrugated plate
  - Result drawn as the ratio of exact result to PFA
  - Plot:
    - Data: H.B. Chan et al
    - Calculations: R. Guérout, J. Lussange, A. Lambrecht
    - Blue \(T=0\)
    - Red \(T=300\text{K}\)


An application to heat transfer calculations

- Heat transfer between two gratings at different temperatures
  - a different observable which can be calculated with the same techniques
  - Large enhancement of the heat conductance wrt plane plates, when corrugation depth is increased, at a constant distance of closest approach (1μm)
  - Proximity approximation predicts a decrease!

**Force in the plane-sphere geometry**

- Force between a plane and a large sphere usually computed using the "Proximity Force Approximation" (PFA)
  - The plane-plane pressure is integrated over the distribution of local inter-plate distance

\[
F_{\text{PFA}} = \int_{L}^{\infty} dA P(L')
\]

- For a plane and a large sphere

\[
F_{\text{PFA}} = 2\pi R \int_{L}^{\infty} dL' P(L')
\]

\[
G_{\text{PFA}} = -\frac{\partial F_{\text{PFA}}}{\partial L} = 2\pi RP(L)
\]

- Lectures R. Decca, H.B. Chan, U. Mohideen

**The plane-sphere geometry beyond PFA**

- General scattering formula

\[
F = k_B T \sum_{\ell} \operatorname{Tr} \left( 1 - R_P e^{-\kappa L} R_S e^{-\kappa L} \right) \xi_n
\]

- Reflection matrices on the plane written as Fresnel amplitudes in the plane waves basis \( \rightarrow R_P \)
- Reflection matrices on the sphere written as Mie amplitudes in the spherical waves basis \( \rightarrow R_S \)
- Transformation from plane to spherical waves (electromagnetic fields)

- We obtain an "exact" multipolar expansion of the energy
  - Spherical waves labeling: \((\ell, m)\), \(|m| \leq \ell\)
  - Sums are truncated for the numerics
  - The results are accurate for \( x \equiv \frac{L}{R} > x_{\text{min}} \), \( x_{\text{min}} \propto \frac{1}{\ell_{\text{max}}} \)


**Effect of geometry for perfect mirrors (T=0)**

- Perfect mirrors described by the plasma model at the limit \( \lambda_p \to 0 \)
- Energy reduced wrt PFA

\[
\rho_E = \frac{E_{PS}}{E_{PS}^\text{PFA}} < 1
\]

- "Exact" electromagnetic result departs from PFA much more rapidly than in previously existing scalar calculations


**Correlation geometry - metallic reflection**

- Perfect or plasma mirrors at null temperature
- Hints on values at low values of \( x \) found through a polynomial extrapolation
- Slope for the plasma model compatible with the experiment of Krause et al

\[
\beta_G \sim -0.21
\]


\[
|\beta_G| \leq 0.4
\]

- Meanwhile slope found for perfect mirrors is not

\[
\beta_G \sim -0.48
\]

**Correlation geometry - temperature**

- Force between plane and spherical perfect reflectors at room or zero temperature.
- Drawn as the ratio of force at \( T \neq 0 \) to force at \( T = 0 \).
- Contribution of thermal photons repulsive at intermediate distances.

\[
F(T) < F(0)
\]

---

**Casimir entropy in the plane-sphere case**

- Casimir entropy at room temperature computed between perfectly reflecting sphere and plane, as a function of separation distance.
- Drawn after division by the volume of the sphere.
- Casimir entropy negative at some distances (for perfect mirrors here).
- Features not seen for perfect plane mirrors.
- Analytical expressions available for small spheres.

---

**Correlation geometry - temperature - dissipation**

- Force between metallic plane and sphere at room temperature.
- Drawn as the ratio of force at \( \gamma = 0 \) (plasma) to force at \( \gamma \neq 0 \) (Drude).
- Plasma and Drude always closer than expected from PFA.
- Ratio at large \( L \) never approaches the factor 2 given by PFA.

---

**Classical limit in the plane-sphere geometry**

- Classical high-temperature limit \( 2\pi k_B T L \gg \hbar c \).
- Free energy dominated by the first Matsubara term.
- Internal energy vanishes \( \mathcal{E} \to 0 \).
- Non null force: entropic origin.
- Known cases:
  - Casimir force vanishes at the classical limit on the 1D line!
  - Casimir force non null at the classical limit in a 3D space!
- Plane-sphere geometry:

\[
\mathcal{F} = \frac{k_B T}{2} \ln \left( 1 - R_P e^{-\kappa L} R_S e^{-\kappa L} \right) \xi = \zeta, \quad \rho(x) = \frac{\mathcal{F}}{\mathcal{F}_{\text{PFA}}}, \quad \mathcal{F}_{\text{PFA}} = \mathcal{F}_{\text{Drude}} - \frac{k_B T \zeta(3)}{8\pi}, \quad \mathcal{F}_{\text{Drude}} = 2\mathcal{F}_{\text{PFA}} \quad x = \frac{L}{R}
\]
Classical limit in the plane-sphere geometry

- Universal (but different) expressions found for Drude and perfect mirrors.
- Very accurate numerical results obtained in this case.


Asymptotic approach to PFA

- We introduce a "slope function" $\beta(x) = 1 + x \beta(x)$.
- It cannot be represented as a Taylor expansion (or a Laurent expansion) in $x$.
- It has a better representation as an expansion in $\ln x$.

$$\beta = a_0 + a_1 y + a_2 y^2 + a_3 y^3$$
$$y \equiv \ln x$$


Dielectric nanospheres above a plate

- Another example: interaction of dielectric nanospheres above a plane (calculation done for $T=0$).
- Problem of interest for neutron physics.


Dielectric nanospheres above a plate

- Absolute value of the energy (eV)
- Problem of interest for neutron physics.


Power laws

- Transition between different regimes represented with "local power laws" versus $1/L$ and $R$
  \[ \nu = -\frac{\partial \ln |\mathcal{E}|}{\partial \ln L}, \quad \mu = \frac{\partial \ln |\mathcal{E}|}{\partial \ln R} \]
- Casimir-Polder at large $L$
  \[ L \rightarrow \infty, \quad \nu \rightarrow 4, \quad \mu \rightarrow 3 \]
- Approaches PFA at small $L$
  \[ L \rightarrow 0, \quad \nu, \mu \rightarrow 1 \]
- Schrödinger problem better behaved for nano-spheres than in the atomic limit $R \rightarrow 0$

Small nano-spheres as large atoms

- Casimir-Polder forces recovered in the atomic limit $R \rightarrow 0$
- Calculations here at $T=0$

- We recover the general expression for an atom (isotropic polarizability $\alpha$) above a plane (isotropic permittivity $\varepsilon_p$) in the dipole approximation

\[ \alpha(i\xi) \propto \frac{\varepsilon_s(i\xi) - 1}{\varepsilon_s(i\xi) + 2} R^3 \]

\[ \varepsilon_{CP}(\xi) \propto \frac{\varepsilon_s(i\xi) - 1}{\varepsilon_s(i\xi) + 2} R^3 \]

Atoms above corrugated plates

- Here, general expression of energy is developed at lowest order in the corrugation amplitudes, for arbitrary corrugation wavevectors
  \[ \delta \mathcal{E}^{\text{corr}} = \int \frac{d^2 k}{4\pi^2} g(k) H(k) e^{i k r} \]
- $H(r)$ corrugation profile
- $H(k)$ Fourier transform

- PFA is recovered as the limit $k \rightarrow 0$
- Deviation from PFA is measured by the factor
  \[ \rho(k) = \frac{g(k)}{g(0)} \]
- Non trivial effect of geometry when $\rho \neq 1$

Example: Rb atom / Si corrugated plate

- Reduction factor for a Rubidium atom above a Silicon corrugated plate, for different distances

This factor turns out to be nearly the same as in the Casimir-Polder limit for a perfect mirror

An analytical expression is obtained in the latter case

\[ \rho = e^{-k L} \left( 1 + k L + \frac{16(k L)^2}{45} + \frac{(k L)^3}{45} \right) \]

Deep corrugations

- Calculations done also for deep gratings made of real materials with arbitrary corrugation depth, period, aspect ratio and distance
- Accounts for diffraction at high orders
- Deviation from PFA predicted in various situations
- Drawn here as the ratio of energy to PFA energy for a Rb atom above the plateau midpoint for a Si grating


Probing non trivial vacua with cold atoms

- Atoms may be considered as local probes of perturbed vacuum
- One can use the possibilities offered by cold atoms to explore non trivial vacuum properties
- Example: monitoring the dynamics of a Bose-Einstein condensate (BEC) above a nano-grooved plate
- BEC acts as a sensor in vacuum
  - Shift of the dipolar oscillation frequency of the BEC


Deep corrugations

BEC acts as a sensor in vacuum

Large corrections to PFA could be measured with BEC acting as a local probe of a "nanostructured quantum vacuum"

D. Dalvit, P.A. Maia Neto, A. Lambrecht, S. Reynaud, PRL 100 (2008) 040405

Probing disorder in quantum vacuum

- Casimir-Polder effect created by a rough surface transfers disorder to quantum vacuum
- The disorder could be revealed by investigating localization properties of a BEC
- BEC acts as a sensor of the "disordered quantum vacuum"
  - experiments already done with a controlled disorder created by laser speckle


Transfering angular momentum to vacuum

- A rotating corrugated plate is able to transfer angular momentum to vacuum
- This can be revealed by the stirring of quantized vortices in an harmonically bound BEC
  - Trapped BEC


- Here PFA predicts no effect at all!
- And pairwise summation (PWS) predicts much smaller magnitude

Again a promising way of investigating the non trivial interplay of geometry on quantum vacuum with cold atoms techniques