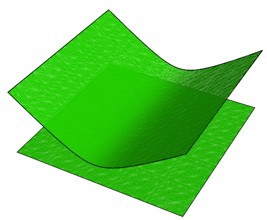
Lecture II: Further Developments in Multiple Scattering

K. A. Milton

I. Cavero-Peláez (Zaragoza), P. Parashar, K.V. Shajesh, and J. Wagner, OU

Klaus Kirsten (BU), Lev Kaplan (Tulane), Iver Brevik and Simen Ellingsen (NTNU)

supported by NSF and DOE (JSF)



I. Green's dyadic formalism

For electromagnetism, we can start from the formalism of Schwinger, which is based on the electric Green's dyadic Γ. of electric fields,

$$\boldsymbol{\Gamma}(\mathbf{r}, t; \mathbf{r}', t') = i\langle \mathsf{T}\{\mathbf{E}(\mathbf{r}, t)\mathbf{E}(\mathbf{r}', t')\}\rangle.$$

Alternatively, we regard the Green's dyadic as the propagator between a polarization source $\mathbf{P}$ and a phenomenological field $\mathbf{E}$ (where $x^\mu = (\mathbf{r}, t)$):

$$\mathbf{E}(x) = \int (dx')\boldsymbol{\Gamma}(x, x) \cdot \mathbf{P}(x').$$

Maxwell's equations

We will only be contemplating static geometries, so it is convenient to consider a specific frequency ω, as introduced through a Fourier transform,

$$\mathbf{\Gamma}(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \mathbf{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega);$$

Maxwell's equations with $\varepsilon(\omega)$ and $\mu(\omega)$ constant in space then read

$$\boldsymbol{\nabla} \times \mathbf{\Gamma} = i\omega \mathbf{\Phi}, \quad \boldsymbol{\nabla} \cdot \mathbf{\Phi} = 0,$$

$$\frac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{\Phi} = -i\omega\varepsilon \mathbf{\Gamma}', \quad \boldsymbol{\nabla} \cdot \mathbf{\Gamma}' = 0,$$

Green's dyadic equations

where we have introduced $\mathbf{\Gamma}' = \mathbf{\Gamma} + \mathbf{1}/\varepsilon$, where the unit dyadic includes a spatial δ function. The two Green's dyadics given here satisfy the following inhomogenous Helmholtz equations,

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Gamma}' = -\frac{1}{\varepsilon}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}),$$

$$(\nabla^2 + \omega^2\varepsilon\mu)\mathbf{\Phi} = i\omega\mu\boldsymbol{\nabla} \times \mathbf{1}.$$

In the following, it will prove more convenient to use

$$\left(\frac{1}{\omega^2\mu}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times -\varepsilon\right)\mathbf{\Gamma} = \mathbf{1}.$$

Variation of action

In the presence of a polarization source, the action is, in symbolic form,

$$W = \frac{1}{2}\int \mathbf{P}\cdot\mathbf{\Gamma}\cdot\mathbf{P},$$

so if we consider the interaction between bodies characterized by particular values of ε and μ, the change in the action due to moving those bodies is

$$\delta W = \frac{1}{2}\int \mathbf{P}\cdot\delta\mathbf{\Gamma}\cdot\mathbf{P} = -\frac{1}{2}\int \mathbf{E}\cdot\delta\mathbf{\Gamma}^{-1}\cdot\mathbf{E},$$

Inverse Green's dyadic

where the symbolic inverse dyadic is

$$\mathbf{\Gamma}^{-1} = \frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times - \varepsilon,$$

that is,

$$\delta\mathbf{\Gamma}\cdot\mathbf{\Gamma}^{-1} = -\mathbf{\Gamma}\cdot\delta\mathbf{\Gamma}^{-1}.$$

Effective sources

By comparing with the iterated source term in the vacuum-to-vacuum persistence amplitude $\exp iW$, we see that an infinitesimal variation of the bodies results in an effective source product,

$$\mathbf{P}(\mathbf{r})\mathbf{P}(\mathbf{r}')\Big|_{\text{eff}} = i\delta\mathbf{\Gamma}^{-1},$$

from which we deduce that

$$\delta W = \frac{i}{2}\text{Tr}\,\mathbf{\Gamma}\cdot\delta\mathbf{\Gamma}^{-1} = -\frac{i}{2}\text{Tr}\,\delta\mathbf{\Gamma}\cdot\mathbf{\Gamma}^{-1} = -\frac{i}{2}\delta\text{Tr}\,\ln\mathbf{\Gamma};$$

trace includes integration over space-time.

Vacuum action

We conclude, by ignoring an integration constant,

$$W = -\frac{i}{2}\text{Tr}\,\ln\boldsymbol{\Gamma}.$$

This is in precise analogy to the expression for scalar fields. Incidentally, note that the first equality above implies for dielectric bodies ($\mu = 1$) the well-known variation form

$$\delta W = -\frac{i}{2}\int\frac{d\omega}{2\pi}\int(d\mathbf{r})\,\delta\varepsilon(\mathbf{r},\omega)\Gamma_{kk}(\mathbf{r},\mathbf{r}';\omega).$$

II. Rederivation of CP formula

Henceforth, let us consider pure dielectrics, that is, set $\mu = 1$. The free Green's dyadic, in the absence of dielectric bodies, satisfies the equation

$$\left[\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -1\right]\boldsymbol{\Gamma}_0 = \mathbf{1},$$

so the equation satisfied by the full Green's dyadic is ($V = \varepsilon - 1$)

$$(\boldsymbol{\Gamma}_0^{-1} - V)\boldsymbol{\Gamma} = \mathbf{1}, \Rightarrow \boldsymbol{\Gamma} = (1 - \boldsymbol{\Gamma}_0 V)^{-1}\boldsymbol{\Gamma}_0.$$

Static energy

From the trace-log formula the energy for a static situation ($W = -\int dt\, E$) relative to the free-space background is

$$E = \frac{i}{2}\text{Tr}\ \ln \boldsymbol{\Gamma}_0^{-1} \cdot \boldsymbol{\Gamma} = -\frac{i}{2}\text{Tr}\ \ln(1 - \boldsymbol{\Gamma}_0 V).$$

The trace here is only over spatial coordinates. We will now consider the interaction between two bodies, with non-overlapping potentials, $V = V_1 + V_2$, where $V_a = \varepsilon_a - 1$ is confined to the interior of body a, $a = 1, 2$.

Weak potentials

Although it is straightforward to proceed to write the interaction between the bodies in terms of scattering operators, for our limited purposes here, we will simply treat the potentials as weak, and retain only the first, bilinear term in the interaction:

$$E_{12} = \frac{i}{2}\,\mathrm{Tr}\,\boldsymbol{\Gamma}_0 V_1 \boldsymbol{\Gamma}_0 V_2.$$

Helmholtz Green's function

Here, as may be verified by direct calculation

$$\mathbf{\Gamma}_0(\mathbf{r},\mathbf{r}') = (\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\zeta^2)G_0(\mathbf{r}-\mathbf{r}'),$$

where the scalar Helmholtz Green's function which satisfies causal or Feynman BC is

$$G_0(\mathbf{r}-\mathbf{r}') = \frac{e^{-|\zeta|R}}{4\pi R}, \quad R = |\mathbf{r}-\mathbf{r}'|,$$

the Fourier transform of the Euclidean Green's function, which obeys the differential equation

$$(-\nabla^2 + \zeta^2)G_0(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \quad \zeta = -i\omega.$$

CP interaction

The interaction between the two potentials:

$$E_{12} = -\frac{1}{2}\int\frac{d\zeta}{2\pi}\int(d\mathbf{r})(d\mathbf{r}')\left[(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\right]^2 \times V_1(\mathbf{r})V_2(\mathbf{r}').$$

$$(\mathbf{\Gamma}_0)_{ij} = \left[-\delta_{ij}(1+|\zeta|R+\zeta^2R^2) + \frac{R_iR_j}{R^2}(3+3|\zeta|R|+\zeta^2R^2)\right]\frac{e^{-|\zeta|R}}{4\pi R^3},$$

CP interaction (cont.)

and then contracting two such factors together gives

$$(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta|R}}{4\pi R}(\nabla_i\nabla_j - \zeta^2\delta_{ij})\frac{e^{-|\zeta|R}}{4\pi R}$$

$$= (6 + 12t + 10t^2 + 4t^3 + 2t^4)\frac{e^{-2t}}{(4\pi R^3)^2},$$

where $t = |\zeta|R$. Inserting this, we obtain for the integral over ζ

Casimir-Polder formula

$$-\frac{1}{64\pi^3 R^7}\int_0^\infty du\, e^{-u}\left(6+6u+\frac{5}{2}u^2+\frac{1}{2}u^3+\frac{1}{8}u^4\right)$$

$$= -\frac{23}{64\pi^3 R^7},$$

or

$$E_{12} = -\frac{23}{(4\pi)^3}\int(d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^7},$$

which is the Casimir-Polder potential. This formula is valid for non-overlapping dielectric bodies.

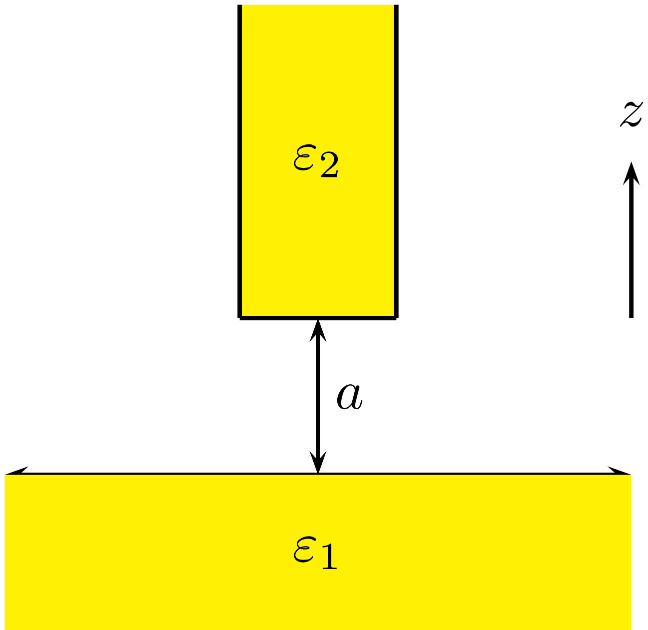
III. Summing van der Waals forces

The (retarded dispersion) van der Waals potential between polarizable molecules is given by

$$V = -\frac{23}{4\pi}\frac{\alpha_1\alpha_2}{r^7}, \quad \alpha = \frac{\varepsilon - 1}{4\pi N}.$$

This allows us to consider in the same vein (electromagnetic) interaction between distinct dilute dielectric bodies of arbitrary shape.

Force between slab/infinite plate



—No correction to Lifshitz formula

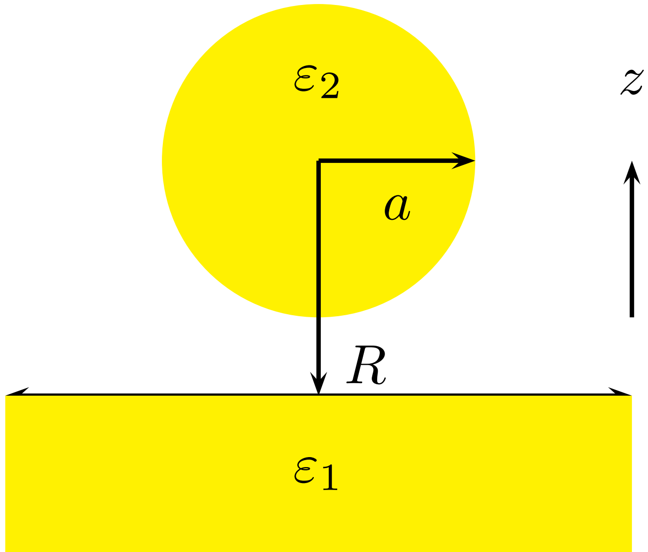
If the cross sectional area of the finite slab is A, the force between the slabs is

$$\frac{F}{A} = -\frac{23}{640\pi^2}\frac{1}{a^4}(\varepsilon_1 - 1)(\varepsilon_2 - 1),$$

the Lifshitz formula for infinite (dilute) slabs.

Note that there is no correction due to the finite area of the slab.

Force between sphere and plate



$$E = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{4\pi a^3/3}{R^4}\frac{1}{(1 - a^2/R^2)^2},$$

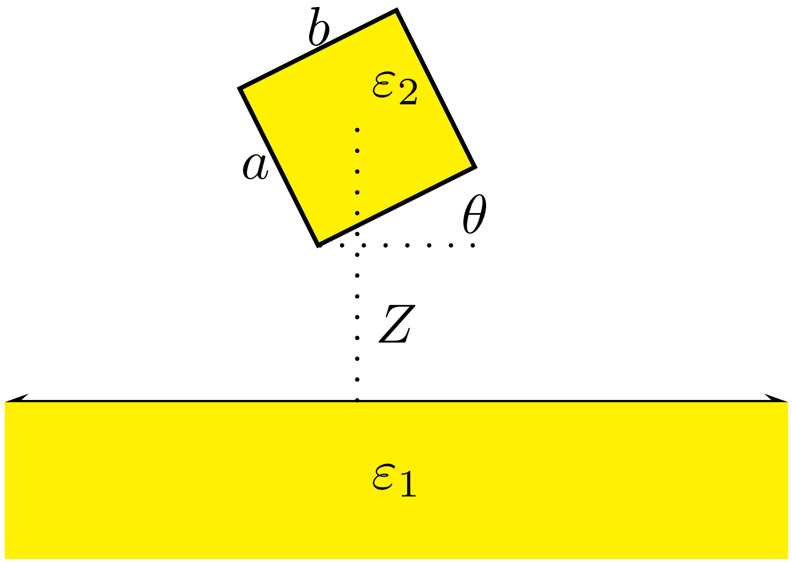
— comparison with PFA

which agrees with the PFA in the short separation limit, $R - a = \delta \ll a$:

$$F_{\mathrm{PFA}} = 2\pi a \mathcal{E}_{\|}(\delta) = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{2\pi a}{3\delta^3},$$

with an exact correction, intermediate between that for scalar 1/2(Dirichlet+Neumann) and electromagnetic perfectly-conducting boundaries.

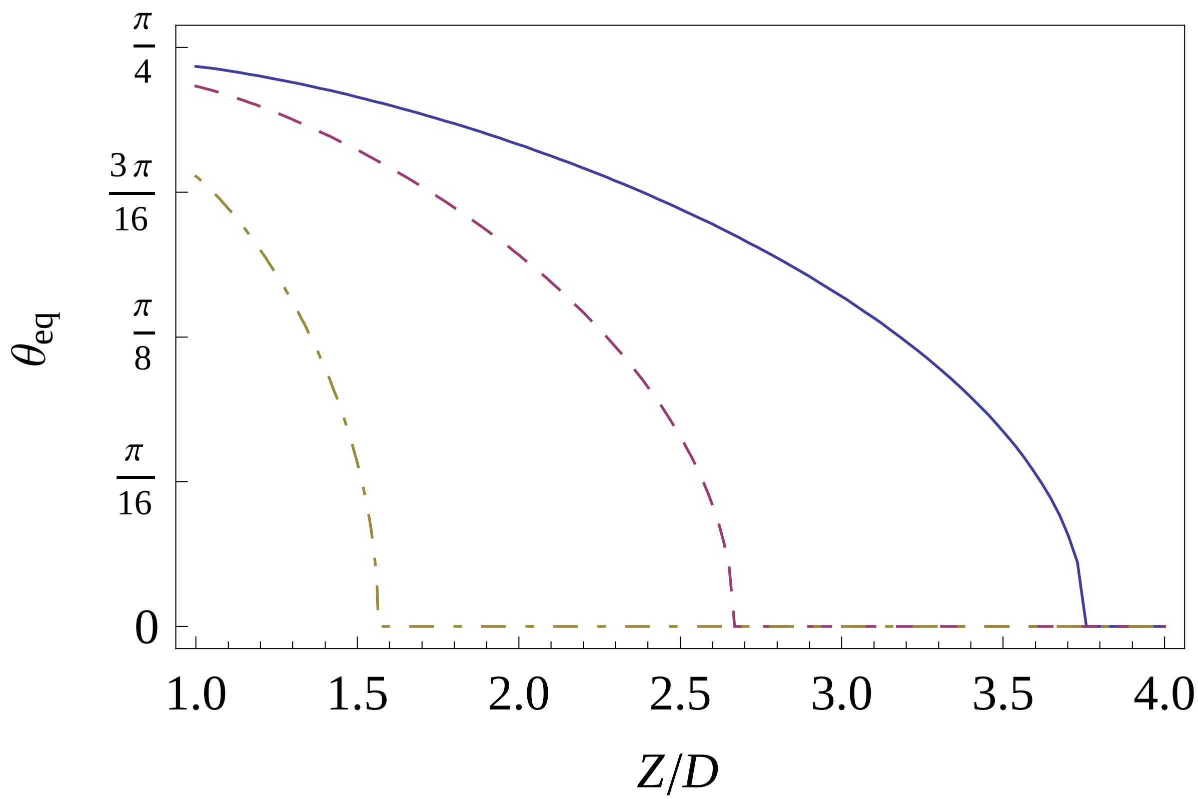
Energy between slab and plate



Torque between slab and plate

Generically, the shorter side wants to align with the plate, which is obvious geometrically, since that (for fixed center of mass position) minimizes the energy. However, if the slab has square cross section, the equilibrium position occurs when a corner is closest to the plate, also obvious geometrically. But if the two sides are close enough in length, a nontrivial equilibrium position between these extremes can occur.

Nontrivial equilibria



Stable equilibria

The stable equilibrium angle of a slab above an infinite plate for given b/a ratios 0.95, 0.9, and 0.7, respectively given by solid, dashed, and dot-dashed lines. For large enough separation, the shorter side wants to face the plate, but for

$$Z < Z_0 = \frac{a}{2}\sqrt{\frac{2a^2 + 5b^2 + \sqrt{9a^4 + 20a^2b^2 + 20b^4}}{5\left(a^2 - b^2\right)}}$$

the equilibrium angle increases, until finally at $Z = D = \sqrt{a^2 + b^2}/2$ the slab touches the plate at an angle $\theta = \arctan b/a$.

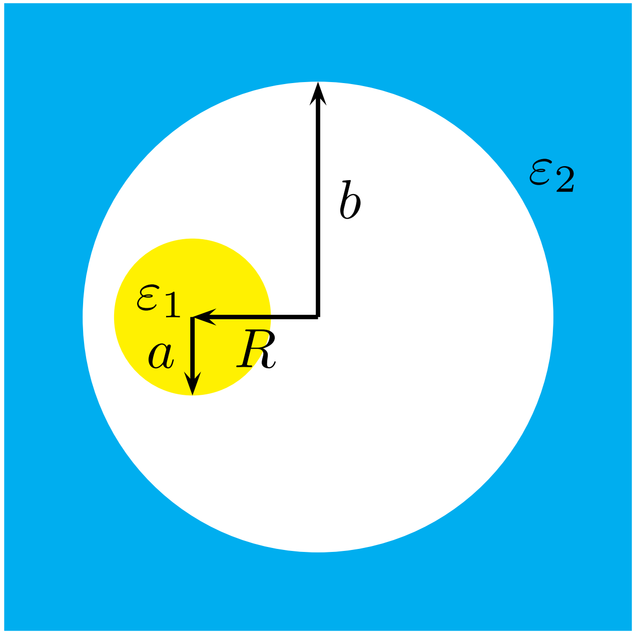
Interaction between ∥ cylinders

$$\frac{E}{L} = -\frac{23}{60\pi}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{a^2b^2}{R^6}$$

$$\times \frac{1 - \frac{1}{2}\left(\frac{a^2+b^2}{R^2}\right) - \frac{1}{2}\left(\frac{a^2-b^2}{R^2}\right)^2}{\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{5/2}}.$$

Eccentric interior cylinder

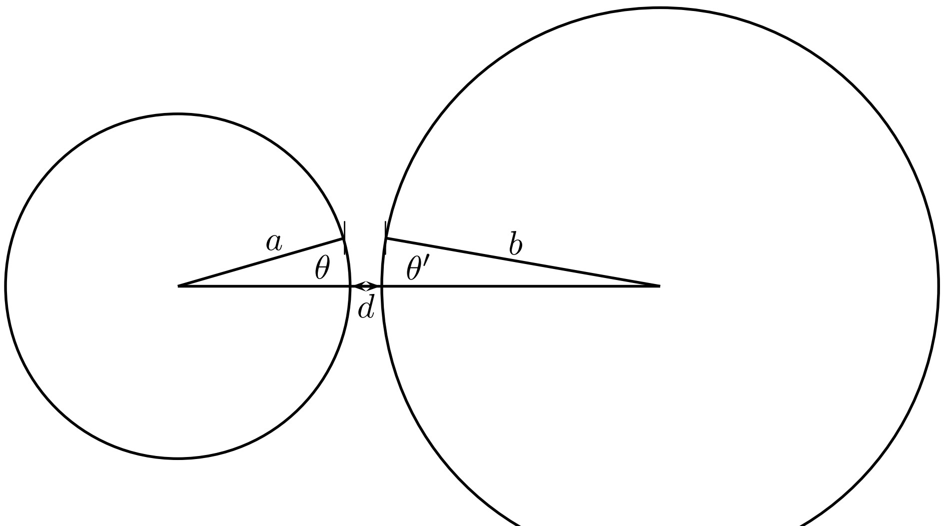
This result can be analytically continued to the case when one dielectric cylinder is entirely inside a hollowed-out cylinder within an infinite dielectric medium.



Interaction between spheres

$$E = -\frac{23}{1920\pi}\frac{(\varepsilon_1-1)(\varepsilon_2-1)}{R}\left\{\ln\left(\frac{1-\left(\frac{a-b}{R}\right)^2}{1-\left(\frac{a+b}{R}\right)^2}\right)\right.$$

$$\left.+\frac{4ab}{R^2}\frac{\frac{a^6-a^4b^2-a^2b^4+b^6}{R^6}-\frac{3a^4-14a^2b^2+3b^4}{R^4}+3\frac{a^2+b^2}{R^2}-1}{\left[\left(1-\left(\frac{a-b}{R}\right)^2\right)\left(1-\left(\frac{a+b}{R}\right)^2\right)\right]^2}\right\}$$



PFA and sphere-plate

This expression, which is rather ugly, may be verified to yield the proximity force theorem:

$$E \to U = -\frac{23}{640\pi}\frac{a(R-a)}{R\delta^2}, \qquad \delta = R - a - b \ll a, b.$$

It also, in the limit $b \to \infty$, $R \to \infty$ with $R - b = Z$ held fixed, reduces to the result for the interaction of a sphere with an infinite plate.

IV. Interaction between ε, μ bodies

Consider material bodies characterized by a permittivity $\varepsilon(\mathbf{r})$ and a permeability $\mu(\mathbf{r})$, so we have corresponding electric and magnetic potentials

$$V_e(\mathbf{r}) = \varepsilon(\mathbf{r}) - 1, \quad V_m(\mathbf{r}) = \mu(\mathbf{r}) - 1.$$

Then the trace-log is ($\boldsymbol{\Phi}_0 = -\frac{1}{\zeta}\boldsymbol{\nabla} \times \boldsymbol{\Gamma}_0$)

$$\begin{aligned}\mathrm{Tr}\,\ln \boldsymbol{\Gamma}\boldsymbol{\Gamma}_0^{-1} &= -\mathrm{Tr}\,\ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_e) - \mathrm{Tr}\,\ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_m)\\ &\quad -\mathrm{Tr}\,\ln(\mathbf{1} + \boldsymbol{\Phi}_0 \mathbf{T}_e \boldsymbol{\Phi}_0 \mathbf{T}_m),\\ \mathbf{T}_{e,m} &= V_{e,m}(\mathbf{1} - \boldsymbol{\Gamma}_0 V_{e,m})^{-1}.\end{aligned}$$

Factorization

If we have *disjoint* electric bodies, the interaction term separates out:

$$\text{Tr}\ \ln\left(\mathbf{1} - \mathbf{\Gamma}_0(V_1 + V_2)\right) = -\text{Tr}\ \ln(\mathbf{1} - \mathbf{\Gamma}_0\mathbf{T}_1) \\ -\text{Tr}\ \ln(\mathbf{1} - \mathbf{\Gamma}_0\mathbf{T}_2) - \text{Tr}\ \ln(\mathbf{1} - \mathbf{\Gamma}_0\mathbf{T}_1\mathbf{\Gamma}_0\mathbf{T}_2),$$

so only the latter term contributes to the interaction energy,

$$E_{\text{int}} = \frac{i}{2}\text{Tr}\ln(\mathbf{1} - \mathbf{\Gamma}_0\mathbf{T}_1\mathbf{\Gamma}_0\mathbf{T}_2).$$

ε-μ Lifshitz force

The same is true if one body is electric and the other magnetic,

$$E_{\text{int}} = -\frac{i}{2}\text{Tr}\ln(1 + \mathbf{\Phi}_0 \mathbf{T}_1^e \mathbf{\Phi}_0 \mathbf{T}_2^m).$$

Using this, it is easy to show that the Lifshitz energy between a parallel dielectric and diamagnetic slabs is

$$E_{\varepsilon\mu} = \frac{1}{16\pi^3}\int d\zeta \int d^2k \Big[\ln\left(1 - r_1 r_2' e^{-2\kappa a}\right) + \ln\left(1 - r_1 r_2' e^{-2\kappa a}\right) \Big]$$

Repulsive Casimir force

where

$$r_i = \frac{\kappa - \kappa_i}{\kappa + \kappa_i}, \qquad r_i' = \frac{\kappa - \kappa_i'}{\kappa + \kappa_i'},$$

with $\kappa^2 = k^2 + \zeta^2$, $\kappa_1^2 = k^2 + \varepsilon\zeta^2$, $\kappa_1' = \kappa_1/\varepsilon$, $\kappa_2^2 = k^2 + \mu\zeta^2$, $\kappa_2' = \kappa_2/\mu$.
This means in the perfect reflecting limit, $\varepsilon \to \infty$, $\mu \to \infty$,

$$E_{\text{Boyer}} = +\frac{7}{8}\frac{\pi^2}{720a^3},$$

we get Boyer's repulsive result.

V. Exact temperature results

The scalar Casimir energy between two weak nonoverlapping potentials $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ at temperature T is

$$E_T = -\frac{T}{32\pi^2}\int (d\mathbf{r})(d\mathbf{r}')V_1(\mathbf{r})V_2(\mathbf{r}')\frac{\coth 2\pi T|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^2}.$$

Exact PFA

The energy between a semitransparent plane and an arbitrarily curved nonintersecting semitransparent surface:

$$E_T = -\frac{\lambda_1\lambda_2 T}{16\pi}\int dS \int_{2\pi T z(S)} dx \frac{\coth x}{x},$$

where the area integral is over the curved surface. This is precisely what one means by the PFA:

$$E_{\rm PFA} = \int dS \mathcal{E}_\parallel(z(S)),$$

as proved by Decca et al.

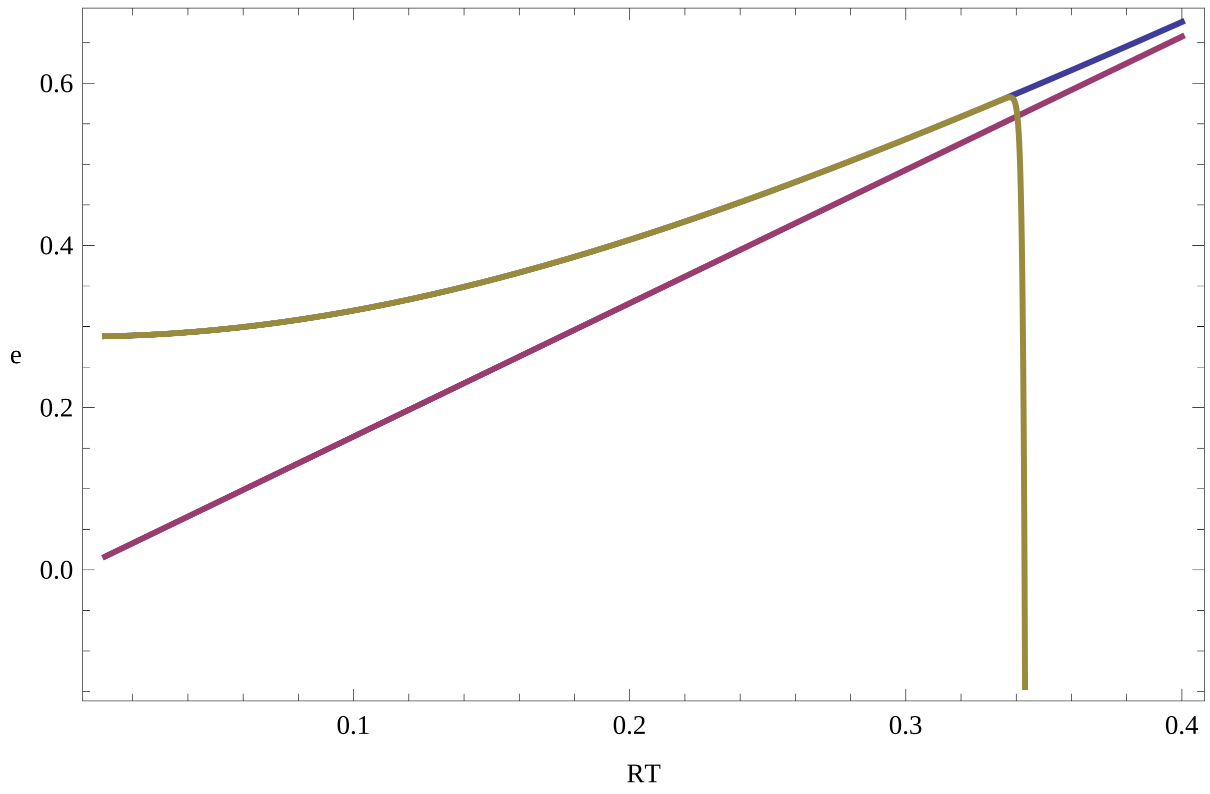
Interaction between ST spheres

$$E_T = -\frac{\lambda_1\lambda_2}{16\pi}\frac{ab}{R}\left\{ \ln\frac{1-(a-b)^2/R^2}{1-(a+b)^2/R^2} \right.$$
$$+ f(2\pi T(R+a+b)) + f(2\pi T(R-a-b))$$
$$\left. - f(2\pi T(R-a+b)) - f(2\pi T(R+a-b)) \right\},$$

where f is obtained from

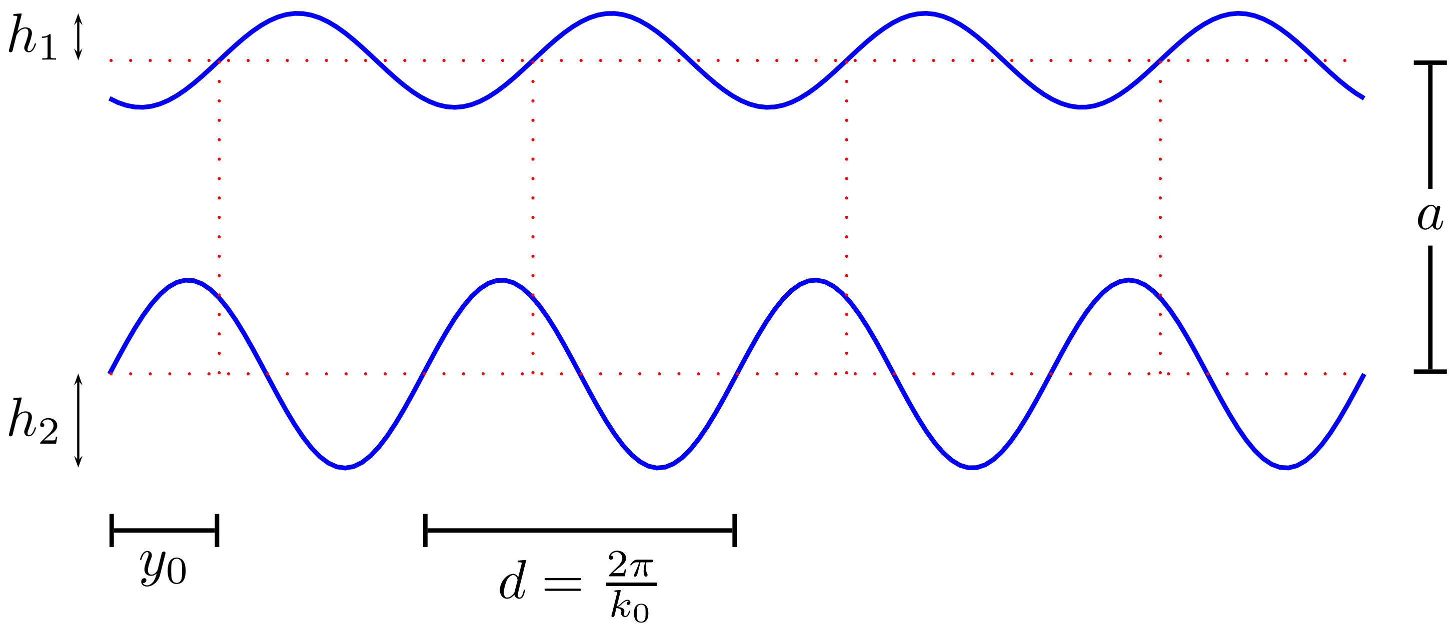
$$y\frac{d^2}{dy^2}f(y) = \coth y - \frac{1}{y}, \qquad f(0) = f'(0) = 0.$$

Numerical results



$a = b = R/4$. Exact, high T, and truncated series expansion.

VI. Noncontact gears

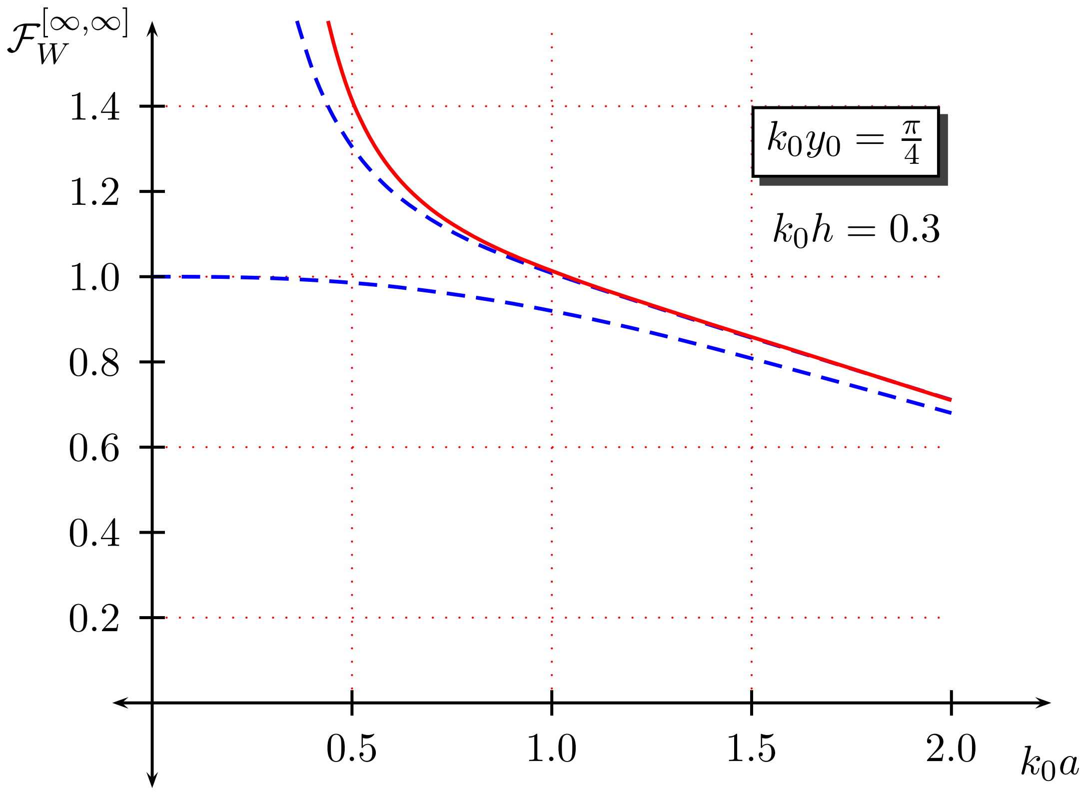


Perturbation theory

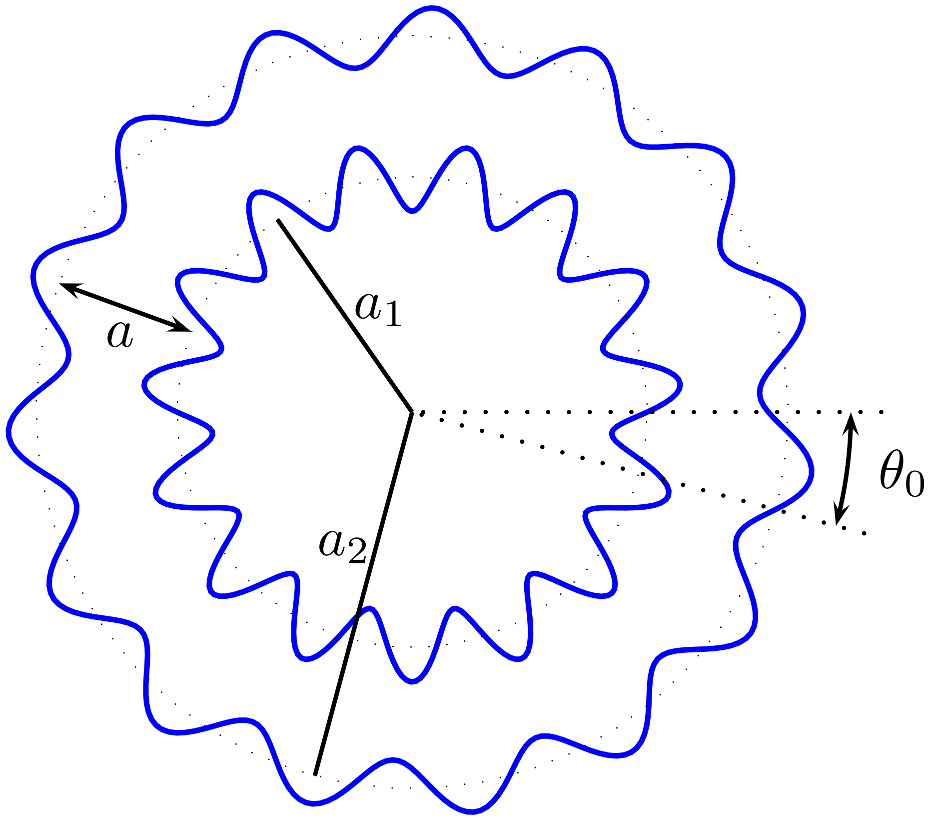
Here we compute the *lateral* force between the offset corrugated plates. The Dirichlet and electromagnetic cases were previously considered by Kardar and Emig, to second order in corrugation amplitudes. We have carried out the calculations to fourth order. In weak coupling we can calculate to all orders, and verify that fourth order is very accurate, provided $k_0 h \ll 1$.

$$\mathcal{F} = \frac{F_{\text{Lat}}}{|F^{(0)}_{\text{Cas}}|(h_1 h_2/a^2) k_0 a \sin(k_0 y)}$$

Weak coupling limit



Concentric corrugated cylinders



Casimir torque per unit area

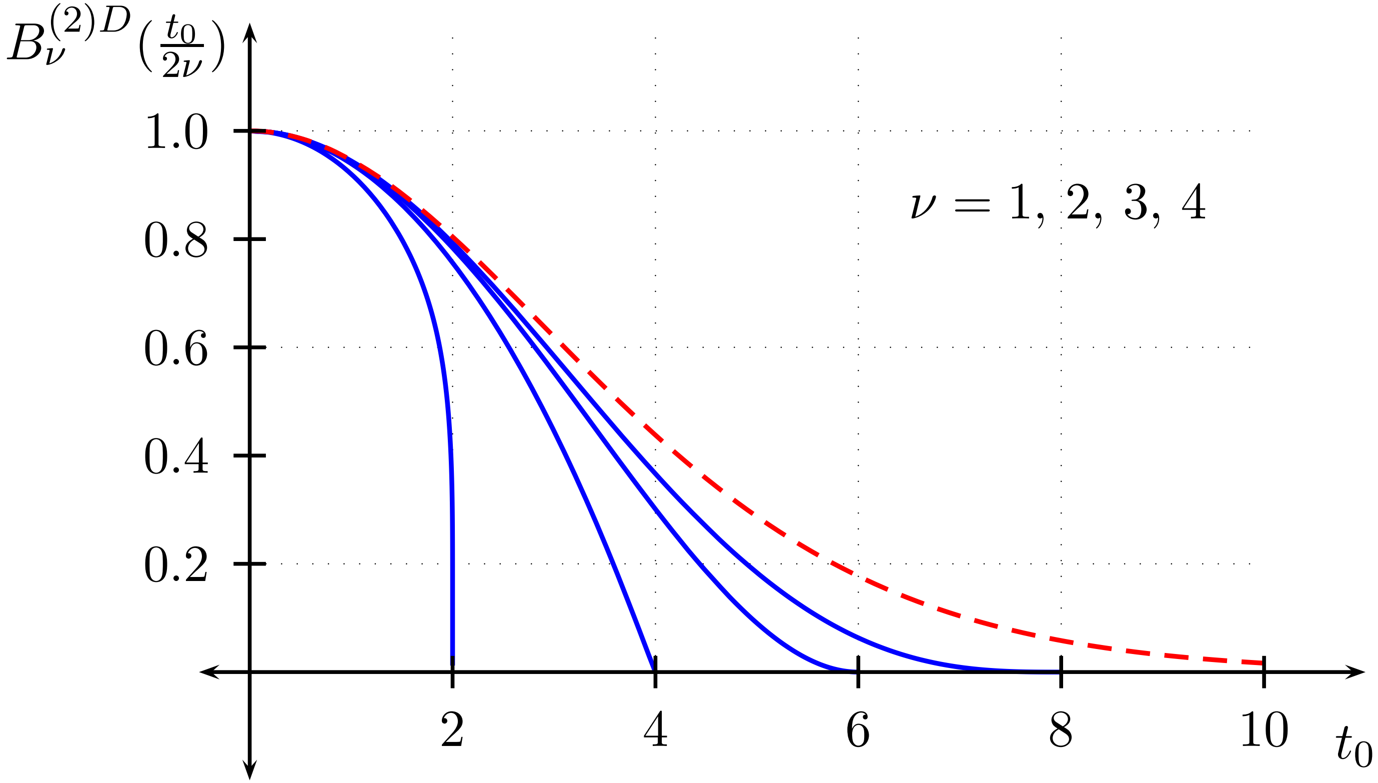
For corrugations given by δ-function potentials with sinusoidal amplitudes:

$$h_1(\theta) = h_1 \sin \nu(\theta + \theta_0),$$
$$h_2(\theta) = h_2 \sin \nu\theta$$

the torque to lowest order in the corrugations in strong coupling (Dirichlet limit) $(\alpha = (a_2 - a_1)/(a_2 + a_1))$

$$\frac{\tau^{(2)D}}{2\pi R L_z} = \nu \sin \nu\theta_0 \frac{\pi^2}{240 a^3} \frac{h_1}{a} \frac{h_2}{a} B_\nu^{(2)D}(\alpha).$$

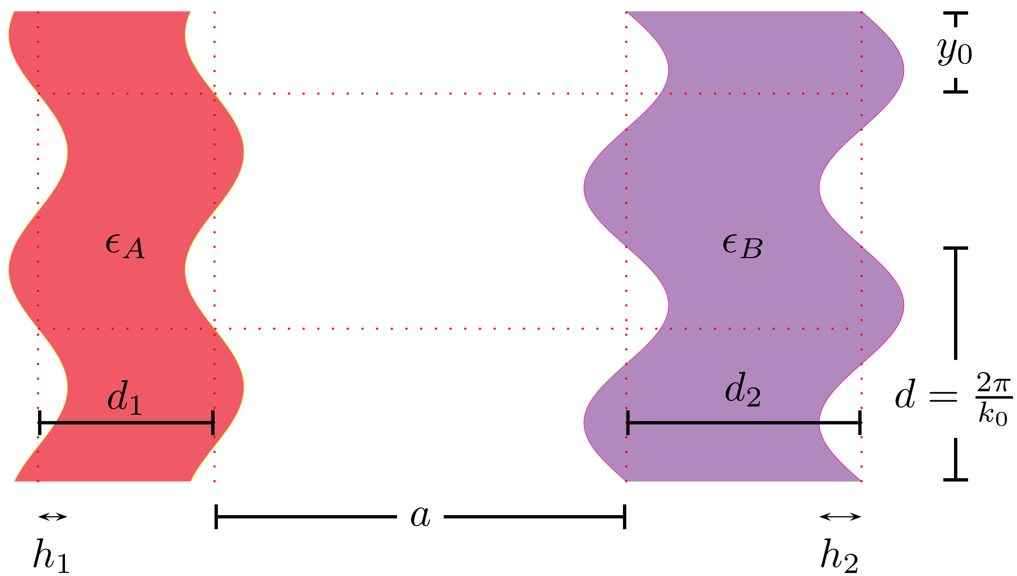
Dirichlet limit of cylindrical gears



A similar result can be found for weak coupling, which, again, has a closed form.

Corrugated Dielectric Slabs

- “Electromagnetic Non-contact Gears: Prelude,” Parashar, Milton, Inés Cavero-Peláez, K.V. Shajesh, arXiv:1001.4105, QFEXT09 proceedings.

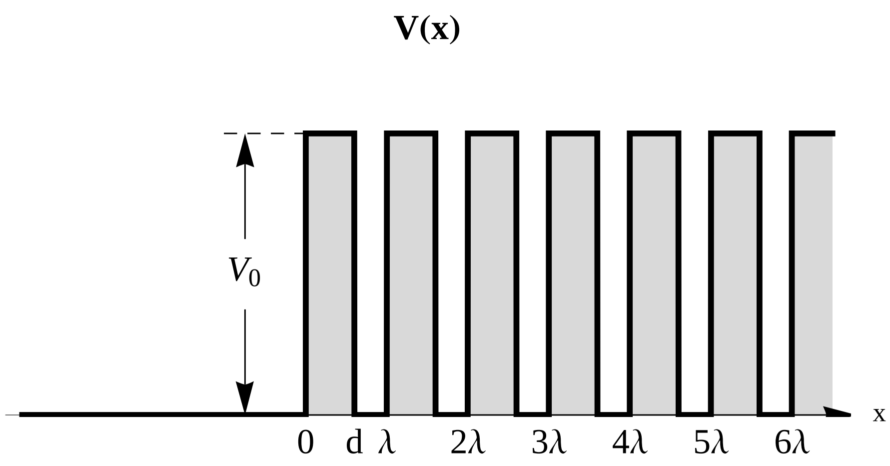


Thick and thin: No difference

Anisotropic atom in front of a semitransparent δ-function plate, versus an anisotropic dielectric slab. to be published in Phys. Rev. D.

VII. Multilayered surfaces

- “Multiple Scattering Casimir Force Calculations: Layered and Corrugated Materials, Wedges, and Casimir-Polder Forces,” Milton, Parashar, Wagner, Inés Cavero-Peláez, arXiv:0910.3215, JVSTB **28**, C4A8-C4A16 (2010).



A semi-infinite array of periodic potentials. The exact CP force between an atom and this array may be calculated.

Green's function

To the right of this array of potentials, the reduced Green's function has the form, in terms the reflection coefficient $\mathcal{R}$ for the array:

$$g(x, x') = \frac{1}{2\kappa} \left(e^{-\kappa|x-x'|} + \mathcal{R} e^{\kappa(x+x')} \right) .$$

(We can actually find the Green's function everywhere, for any piecewise continuous potential. This is described in detail in Jef Wagner's thesis.)

Reflection coefficient

The array reflection coeffient may be readily expressed in terms of the reflection and transmission coefficients for a single potential:

$$\mathcal{R} = R + Te^{-\kappa a}\mathcal{R}e^{-\kappa a}(1 - Re^{-2\kappa a}\mathcal{R})^{-1},$$

where a is the distance between the potentials, the result of summing multiple reflections, $\longrightarrow$

$$\mathcal{R} = \frac{1}{2R}\Big[e^{2\kappa a} + R^2 - T^2 - \sqrt{(e^{2\kappa a} - R^2 - T^2)^2 - 4R^2T^2}\Big].$$

Dielectric slabs

If the potentials consist of dielectric slabs, with dielectric constant ε and thickness d, the TE reflection and transmission coefficients for a single slab are ($\kappa' = \sqrt{\varepsilon\zeta^2 + k^2}$)

$$R^{\mathrm{TE}} = \frac{e^{2\kappa' d} - 1}{\left(\frac{1+\kappa'/\kappa}{1-\kappa'/\kappa}\right) e^{2\kappa' d} - \left(\frac{1-\kappa'/\kappa}{1+\kappa'/\kappa}\right)},$$

$$T^{\mathrm{TE}} = \frac{4(\kappa'/\kappa)e^{\kappa' d}}{(1+\kappa'/\kappa)^2 e^{2\kappa' d} - (1-\kappa'/\kappa)^2}.$$

TM: except in the exponents, $\kappa' \to \kappa'/\varepsilon$.

CP Force

Consider an atom, of polarizability $\alpha(\omega)$, a distance Z to the left of the array. The Casimir-Polder energy is

$$E = -\int d\zeta \int \frac{d^2k}{(2\pi)^2}\alpha(i\zeta) g_{kk}(Z,Z),$$

where apart from an irrelevant constant the trace is

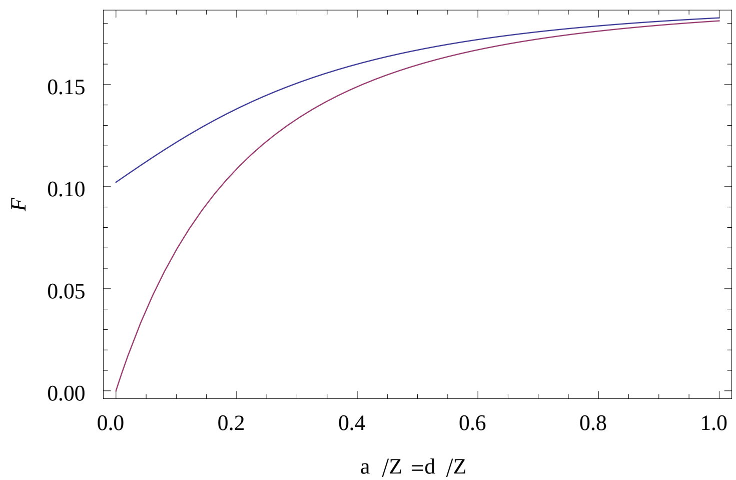
$$g_{kk}(Z,Z) \to \frac{1}{2\kappa}\left[-\zeta^2\mathcal{R}^{\mathrm{TE}} + (\zeta^2+2k^2)\mathcal{R}^{\mathrm{TM}}\right] e^{-2\kappa|Z|}.$$

Numerical Results ($\varepsilon = 2$)

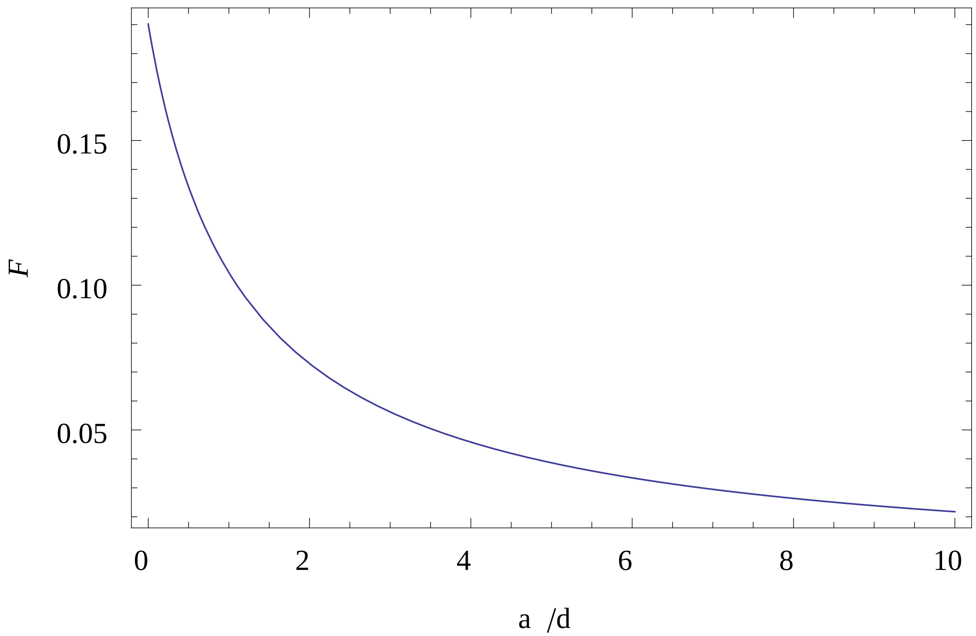
For example, in the static limit, where we disregard the frequency dependence of the polarizability,

$$E = -\frac{\alpha(0)}{2\pi}\frac{1}{Z^4}F(a/Z, d/Z).$$

This is compared with the single slab result:

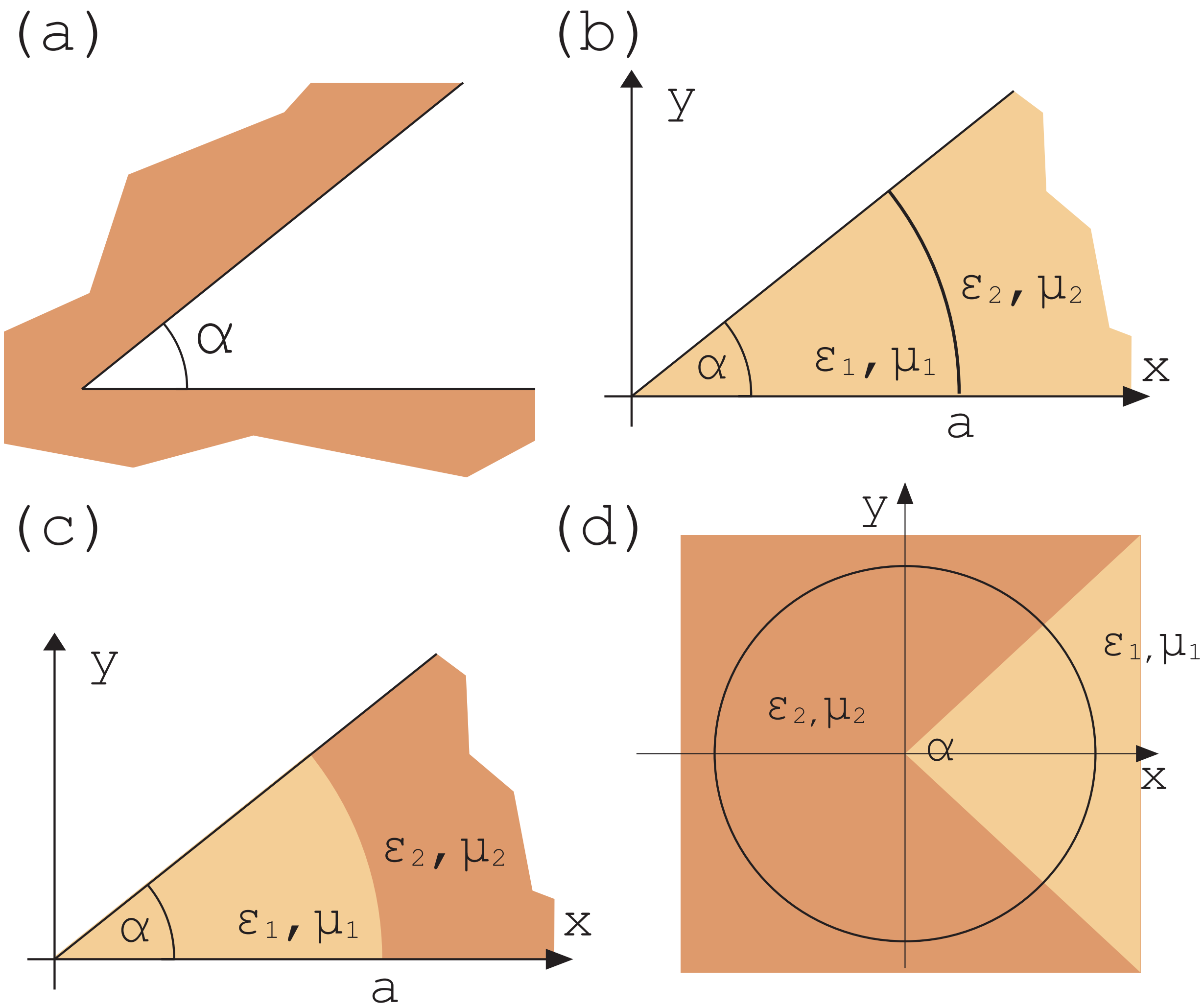


$Z \to \infty$ limit ($\varepsilon = 2$)



- When $a/d \to 0$ we recover the bulk limit.

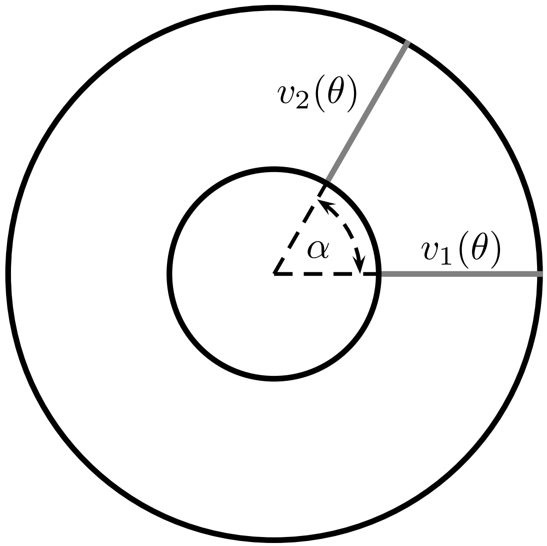
VIII. Wedge generalizes cylinder



Annular Piston

- "Scalar Casimir Energies for Separable Coordinate Systems: Application to Semi-transparent Planes in an Annulus," Wagner, Milton, K. Kirsten, arXiv:0912.2374, in the proceedings of QFEXT09
- "Casimir Effect for a Semitransparent Wedge and an Annular Piston," Milton, Wagner, Klaus Kirsten, arXiv:0911.1123, Phys. Rev. D **80**, 125028 (2009)

Annular piston–ST plates



We use multiple scattering in the angular coordinates, and an eigenvalue condition in the radial coordinates—equally well solvable with radial Green's functions, but generalizable.

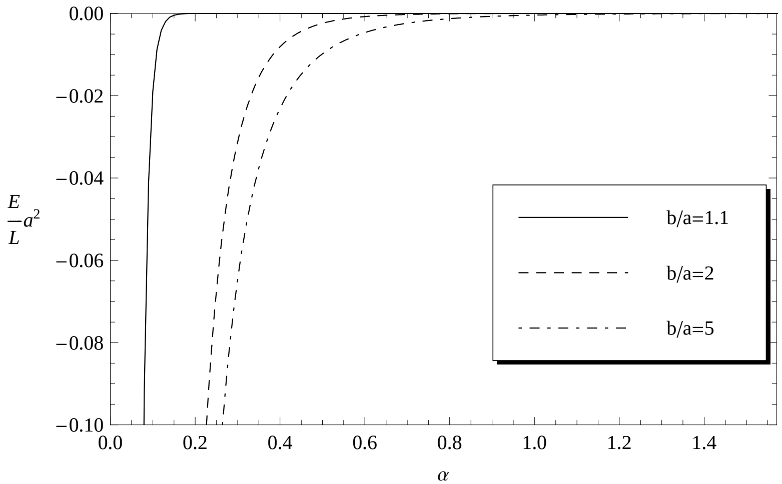
Energy for annular Casimir piston

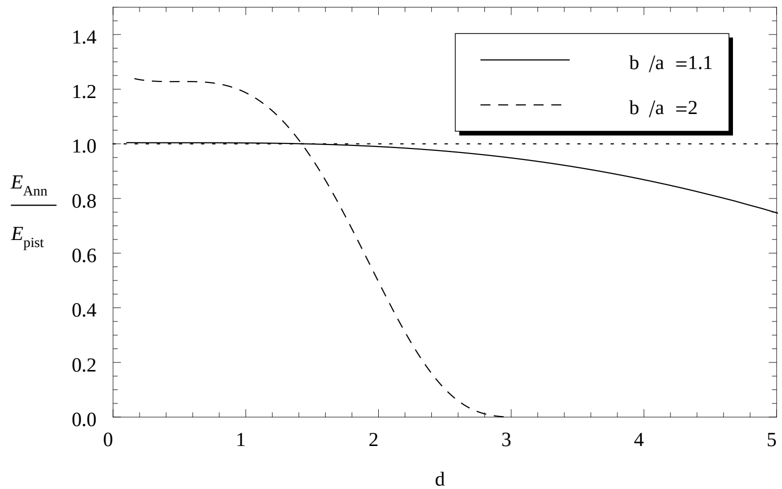
Using the argument principle to determine the angular eigenvalues, we get the following expression for the energy for an annular Casimir piston: ($\tilde{I}_\nu = \frac{1}{2}(I_\nu + I_{-\nu})$) $\mathcal{E} =$

$$\int_0^\infty \frac{\kappa d\kappa}{8\pi^2 i} \int_\gamma d\eta \frac{\partial}{\partial \eta} \ln \left[K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa b) - \tilde{I}_{i\eta}(\kappa a) K_{i\eta}(\kappa b) \right]$$

$$\times \ln \left(1 - \frac{\lambda_1 \lambda_2 \cosh^2 \eta(\pi - \alpha) / \cosh^2 \eta\pi}{(2\eta \tanh \eta\pi + \lambda_1)(2\eta \tanh \eta\pi + \lambda_2)} \right).$$

MS calculation for annular piston





Energy/length for annular piston as function of angle (left), and compared to energy/length for rectangular piston (right). Here $d = \frac{b+a}{2}\sin\frac{\alpha}{2}$, and plateaus may be understood from the PFA,

$$\frac{\mathcal{E}_{\rm PFA}}{\mathcal{E}_\parallel} = \frac{1}{16}\frac{b^2}{a^2}\left(1+\frac{a}{b}\right)^4.$$

Summary for Lecture II

- Derived Green's dyadic formulism, including CP interaction
- Presented results for summing retarded CP force between dilute bodies
- Considered permeable as well as dielectric bodies
- Considered T dependence for weak coupling
- Investigated non-contact gears, and thin conductors
- CP interactions with multilayered surfaces
- Wedges and annular pistons