# Lecture I: Exact Multiple Scattering Results for Weak-Coupling Forces Between Bodies

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# I. Multiple Scattering Technique

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we first restrict attention to a massless scalar field)($\tau$ is the “infinite” time that the configuration exists) [Schwinger, 1975]

$$E = \frac{i}{2\tau}\text{Tr}\ln G \to \frac{i}{2\tau}\text{Tr}\ln G G_0^{-1},$$

where $G$ ($G_0$) is the Green’s function,

$$(-\partial^2 + V)G = 1, \quad +\text{BC}, \quad -\partial^2 G_0 = 1.$$

# Derivation of MS formula

Following Schwinger, we start from the vacuum persistence amplitude in terms of sources,

$$\langle 0_+|0_-\rangle^K = e^{iW[K]}$$

$$W[K] = \frac{1}{2}\int (dx)(dx')K(x)G(x,x')K(x'),$$

where $G$ is the Green's function in the presence of the background potential. Effective field:

$$\phi(x) = \int (dx')G(x,x')K(x').$$

# Alteration of geometry

If the geometry of the region is altered slightly, as through moving one of the bounding surfaces, the vacuum amplitude is altered:

$$\delta W[K] = \frac{1}{2}\int (dx)(dx')K(x)\delta G(x,x')K(x')$$
$$= -\frac{1}{2}\int (dx)(dx')\phi(x)\delta G^{-1}(x,x')\phi(x),$$

which uses the fact that $GG^{-1} = 1$.

# Effective source

Compare this with the two-particle emission term,

$$e^{iW[K]} = \exp i \int (dx)[K(x)\phi(x) + \mathcal{L}$$
$$= \cdots + \frac{1}{2}\left[i \int (dx) K(x)\phi(x)\right]^2 .$$

Effective 2-particle source:

$$iK(x)K(x')\Big|_{\text{eff}} = -\delta G^{-1}(x, x').$$

# $\mathbf{Tr}\,\ln$ formula

Then we get (using matrix notation)

$$\delta W = \frac{i}{2}\mathsf{Tr}\, G\delta G^{-1} = -\frac{i}{2}\mathsf{Tr}\,\delta G G^{-1}.$$

From this,

$$\delta W = -\frac{i}{2}\delta \mathsf{Tr}\,\ln G.$$

or with a static configuration, $W = -E\tau$, where $\tau$ is the (infinite) time the configuration exists,

$$E = \frac{i}{2\tau}\mathsf{Tr}\,\ln G.$$

# Equivalent forms of QVE

$$E = \frac{i}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathsf{Tr}\ln\mathcal{G},$$

$$G(x.x') = \int_{-\infty}^{\infty} e^{-i\omega(t-t')}\mathcal{G}(\mathbf{r},\mathbf{r}';\omega).$$

By integration by parts, the energy can be cast into another well-known form:

$$E = -i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2\mathsf{Tr}\,\mathcal{G}.$$

# $T$-matrix

Now we define the $T$-matrix,

$$T = S - 1 = V(1 + G_0 V)^{-1}.$$

If the potential has two disjoint parts, $V = V_1 + V_2$ it is easy to derive the interaction between the two bodies (potentials):

$$\begin{aligned} E_{12} &= -\frac{i}{2\tau} \mathrm{Tr} \ln(1 - G_0 T_1 G_0 T_2) \\ &= -\frac{i}{2\tau} \mathrm{Tr} \ln(1 - V_1 G_1 V_2 G_2), \end{aligned}$$

where $G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$

# II. $2 + 1$ dimensions

To proceed to apply this method to general bodies, we use an even older technique, the multipole expansion. Let's illustrate this with a $2 + 1$ dimensional version, which allows us to describe cylinders with parallel axes. We seek an expansion of the free Green's function

$$G_0(\mathbf{R} + \mathbf{r}' - \mathbf{r}) = \frac{e^{i|\omega||\mathbf{r}-\mathbf{R}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{R} - \mathbf{r}'|}$$

$$= \int \frac{dk_z}{2\pi} e^{ik_z(z-Z-z')} g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp),$$

# Reduced Green's function

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \int \frac{(d^2k_\perp)}{(2\pi)^2} \frac{e^{-i\mathbf{k}_\perp\cdot\mathbf{R}_\perp} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)}}{k_\perp^2 + k_z^2 + \zeta^2}.$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp - \mathbf{R}_\perp - \mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r) e^{im\phi} I'_m(\kappa r') e^{-im'\phi'}$$
$$\times \tilde{g}^0_{m,m'}(\kappa R), \quad \kappa^2 = k_z^2 + \zeta^2.$$

# Expression for $g^0_{m,m'}$

By Fourier transforming, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-im\phi} e^{ikr\cos\phi},$$

we easily find

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{1}{2\pi}\int \frac{dk\,k}{k^2+\kappa^2} J_{m-m'}(kR)\frac{J_m(kr)J_m(kr')}{I_m(\kappa r)I_m(\kappa r')}$$
$$= \frac{(-1)^{m'}}{2\pi} K_{m-m'}(\kappa R).$$

# Discrete matrix realization

Thus we can derive an expression for the interaction between two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2}\int d\zeta\, dk_z \ln\det\left(1 - \tilde{g}^0\tilde{T}_1\tilde{g}^{0\top}\tilde{T}_2\right),$$

where the $\tilde{T}$ matrix elements are given by

$$\tilde{T}_{mm'} = \int dr\, r\, d\phi \int dr'\, r'\, d\phi' I_m(\kappa r)e^{-im\phi} I_{m'}(\kappa r')e^{im'\phi'} \times T(r,\phi;r',\phi').$$

# Interaction between cylinders

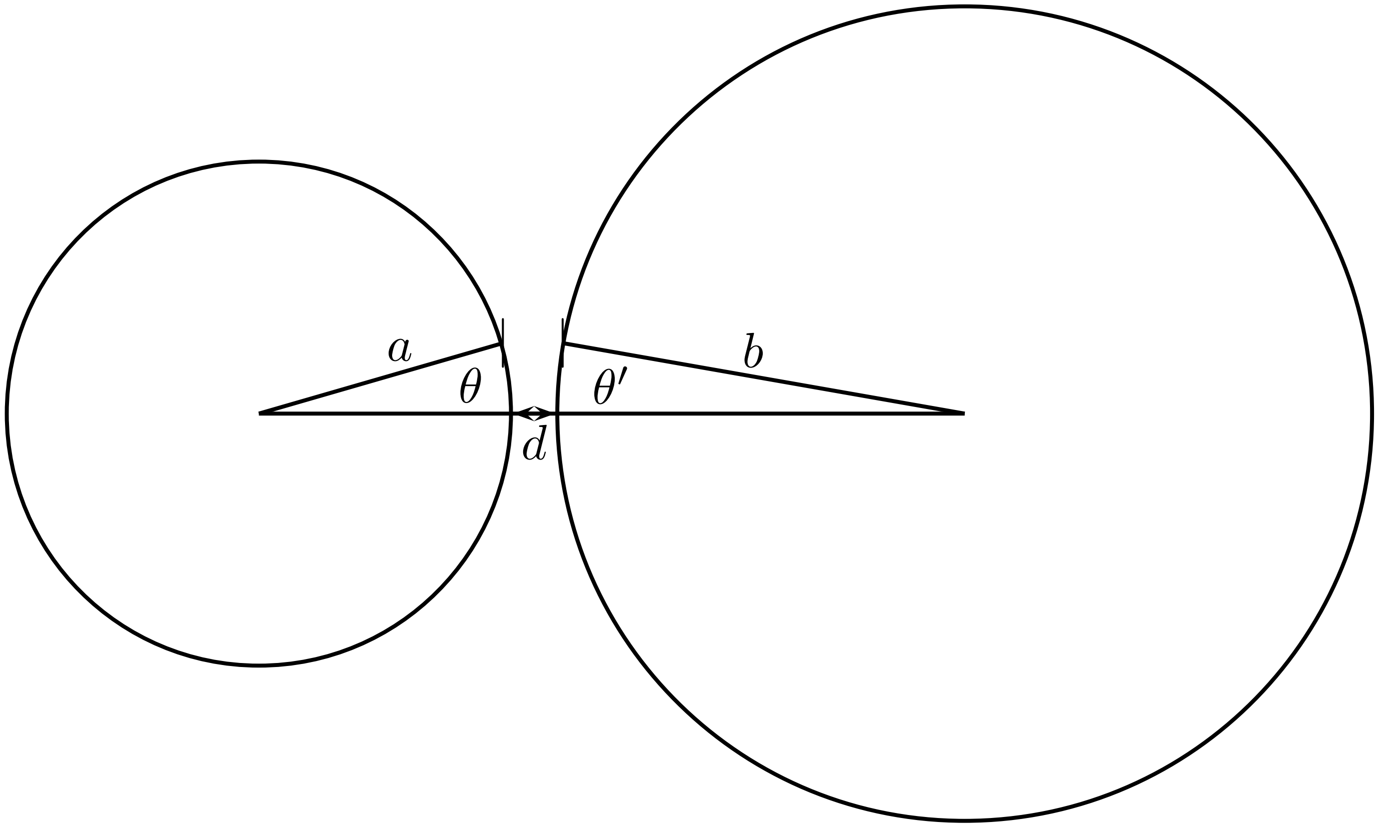



Figure 1: Geometry of two cylinders (or two spheres) with radii $a$ and $b$, respectively, and distances between their centers of $R > a + b$.

# Semitransparent cylinders

Consider two parallel semitransparent cylinders, of radii $a$ and $b$, respectively, lying outside each other, described by the potentials

$$V_1 = \lambda_1 \delta(r - a), \quad V_2 = \lambda_2 \delta(r' - b),$$

with the separation between the centers $R$ satisfying $R > a + b$. It is easy to work out the scattering matrix in this situation,

$$(t_1)_{mm'} = 2\pi\lambda_1 a \delta_{mm'} \frac{I_m^2(\kappa a)}{1 + \lambda_1 a I_m(\kappa a) K_m(\kappa a)}.$$

# Cylinder interaction

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty d\kappa\,\kappa\,\mathrm{tr}\ln(1-A),$$

where $A = B(a)B(b)$, in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R)\frac{\lambda_1 a I^2_{m'}(\kappa a)}{1+\lambda_1 a I_{m'}(\kappa a)K_{m'}(\kappa a)}.$$

# Weak-coupling

In weak coupling, the formula for the interaction energy between two cylinders is

$$\mathfrak{E} = -\frac{\lambda_1\lambda_2 ab}{4\pi R^2}\sum_{m,m'=-\infty}^{\infty}\int_0^\infty dx\, x\, K^2_{m+m'}(x)$$
$$\times I^2_m(xa/R)I^2_{m'}(xb/R).$$

# Power series expansion

One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$:

$$I_m^2(x) = \left(\frac{x}{2}\right)^{2|m|} \sum_{n=0}^{\infty} Z_{|m|,n} \left(\frac{x}{2}\right)^{2n},$$

where the coefficients $Z_{m,n}$ are

$$Z_{m,n} = \frac{2^{2(m+n)}\,\Gamma\left(m+n+\frac{1}{2}\right)}{\sqrt{\pi}\;n!\;(2m+n)!\;\Gamma(m+n+1)}.$$

# Closed form result

In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2}\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n} P_n(\mu),$$

where $\mu = b/a$, and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^{n}\binom{n}{k}^2 \mu^{2k}.$$

# Closed form result (cont.)

Remarkably, it is possible to perform the sums, so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2}\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{-1/2}.$$

# PFA

We note that in the limit $R - a - b = d \to 0$, $d$ being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case

$$U(d) = -\frac{\lambda_1 \lambda_2}{32\pi} \sqrt{\frac{2ab}{R}} \frac{1}{d^{1/2}}, \qquad d \ll a, b.$$

The rate of approach is given by

$$\frac{\mathfrak{E}}{U} \approx 1 - \frac{1 + \mu + \mu^2}{4\mu} \frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R-a)} \frac{d}{R}.$$

# $a = b$

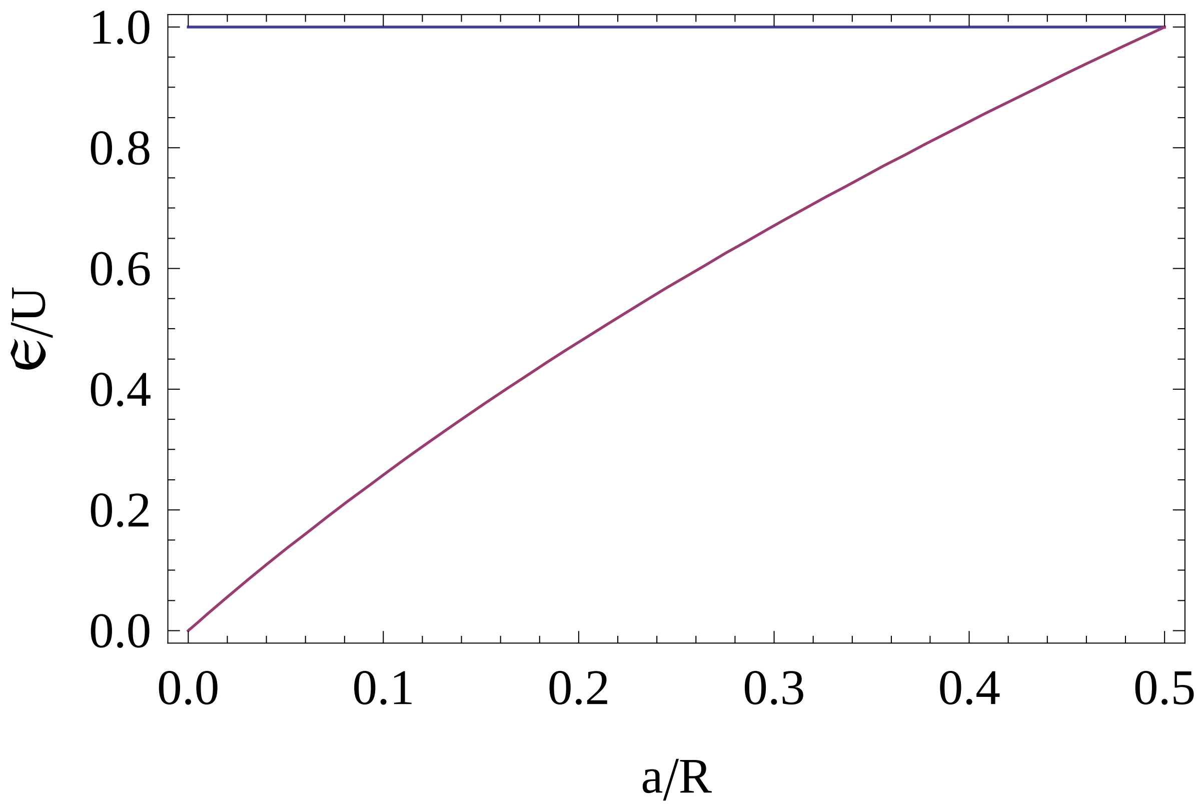



Figure 2: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

# $b/a = 99$

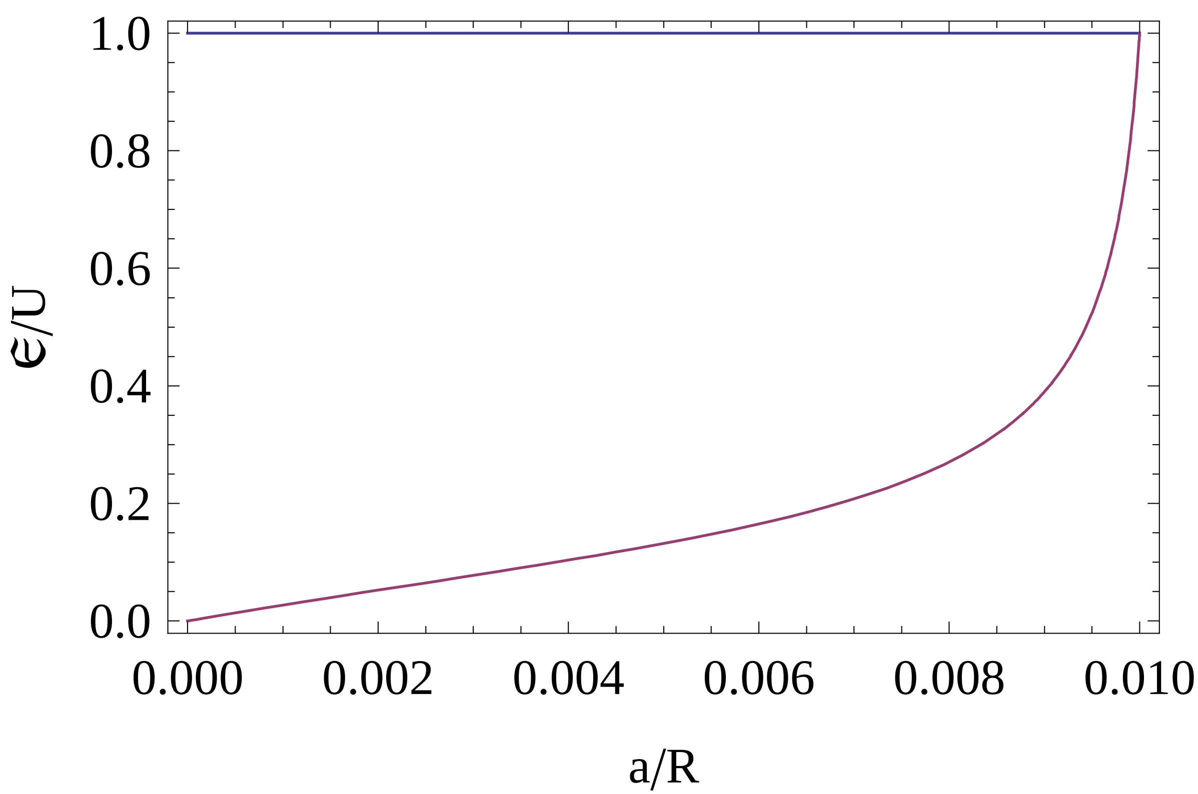



Figure 3: Plotted is the ratio of the exact interaction energy of two weakly-coupled cylinders to the proximity force approximation

# Cylinder/plane interaction

By the method of images, we can find the interaction between semitransparent cylinder and a Dirichlet plane is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty \kappa\, d\kappa\, \text{tr}\, \ln(1 - B(a)),$$

where $B(a)$ is given above. In the strong-coupling limit this result agrees with that given by Bordag, because

$$\text{tr}\, B^s = \text{tr}\, \tilde{B}^s, \quad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)} K_{m+m'}(\kappa R) I_{m'}(\kappa a).$$

# Exact cylinder/plane energy

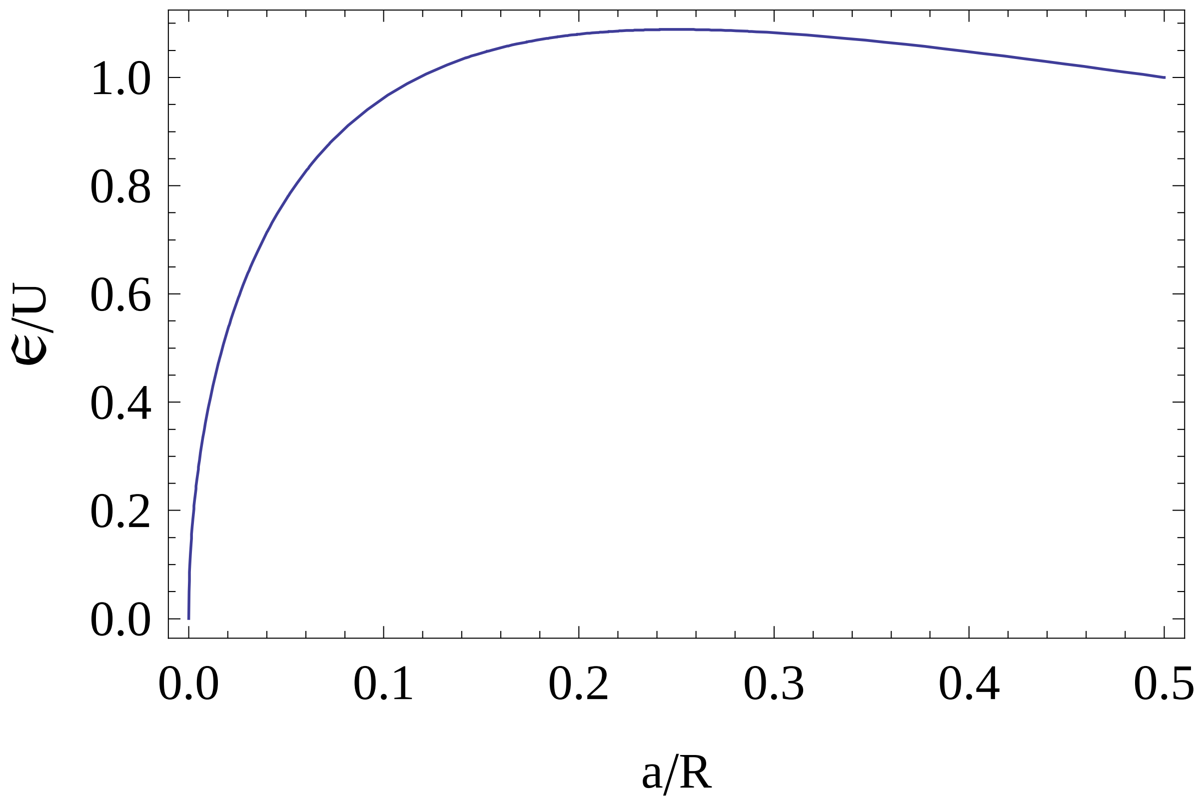
In exactly the same way, we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius $a$ separated by a distance $R/2$. The result is again quite simple:

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\left[1-\left(\frac{2a}{R}\right)^2\right]^{-3/2}.$$

In the limit as $d \to 0$, this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi}\frac{\sqrt{2a}}{d^{3/2}}.$$

# Comparison of PFA and exact



# III. 3-dimensional formalism

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \sum_{lm,l'm'} j_l(i|\zeta|r)j_{l'}(i|\zeta|r')Y^*_{lm}(\hat{\mathbf{r}})Y_{l'm'}(\hat{\mathbf{r}}') \times g_{lm,l'm'}(\mathbf{R}).$$

# Reduced Green's function

The reduced Green's function can be written in the form

$$g^0_{lm,l'm'}(\mathbf{R}) = (4\pi)^2 i^{l'-l} \int \frac{(d\mathbf{k})}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2+\zeta^2} \frac{j_l(kr)j_{l'}(kr')}{j_l(i|\zeta|r)j_{l'}(i|\zeta|r')}$$
$$\times Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}).$$

# Plane-wave expansion

Now we use the plane-wave expansion once again, this time for $e^{i\mathbf{k}\cdot\mathbf{R}}$,

$$e^{i\mathbf{k}\cdot\mathbf{R}} = 4\pi \sum_{l''m''} i^{l''} j_{l''}(kR) Y_{l''m''}(\hat{\mathbf{R}}) Y^*_{l''m''}(\hat{\mathbf{k}}),$$

so now we encounter something new, an integral over three spherical harmonics,

$$\int d\hat{\mathbf{k}} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y^*_{l''m''}(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''},$$

# Wigner coefficients

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''}\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}.$$

The three-$j$ symbols (Wigner coefficients) here vanish unless $l + l' + l''$ is even.

# Reduced Green's function

This fact is crucial, since because of it we can follow the previous method of writing $j_{l''}(kR)$ in terms of Hankel functions of the first and second kind, using the reflection property of the latter, $h^{(2)}_{l''}(kR) = (-1)^{l''} h^{(1)}_{l''}(-kR)$, and then extending the $k$ integral over the entire real axis to a contour integral closed in the upper half plane.

$$g^0_{lm,l'm'}(\mathbf{R}) = 4\pi i^{l'-l}\sqrt{\frac{2|\zeta|}{\pi R}}\sum_{l''m''} C_{lm,l'm',l''m''}$$

$$\times K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}).$$

# Casimir interaction of spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1 \delta(r-a), \quad V_2(r') = \lambda_2 \delta(r'-b),$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r}, \mathbf{r}') = \frac{\lambda_1}{a^2} \delta(r-a) \delta(r'-a) \times \sum_{lm} \frac{Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')}{1 + \lambda_1 a K_{l+1/2}(|\zeta| a) I_{l+1/2}(|\zeta| a)},$$

# Scattering matrix element

and then the harmonic transform is very similar to that seen for the cylinder, ($k = i|\zeta|$)

$$(t_1)_{lm,l'm'} = \int (d\mathbf{r})(d\mathbf{r}') j_l(kr) Y^*_{lm}(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r}, \mathbf{r}'$$

$$= \delta_{ll'}\delta_{mm'}(-1)^l \frac{\lambda_1 a\pi}{2|\zeta|} \frac{I^2_{l+1/2}(|\zeta|a)}{1 + \lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}.$$

# Interaction energy

Let us suppose that the two spheres lie along the $z$-axis, that is, $\mathbf{R} = R\hat{\mathbf{z}}$. Then we can simplify the expression for the energy somewhat by using $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$. The formula for the energy of interaction becomes

$$E = \frac{1}{2\pi}\int_0^\infty d\zeta\, \mathrm{tr}\ln(1-A),$$

where the matrix

$$A_{lm,l'm'} = \delta_{m,m'}\sum_{l''} B_{ll''m}(a)B_{l''l'm}(b)$$

# $B_{ll'm}$

$$B_{ll'm}(a) = \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} i^{-l+l'} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (2l''+1)$$

$$\times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(\zeta R) \lambda_1 a I^2_{l'+1/2}(\zeta a)}{1 + \lambda_1 a I_{l'+1/2}(\zeta a) K_{l'+1/2}(\zeta a)}$$

Note that the phase always cancels in the trace.

# Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$\sum_{m=-l}^{l}\begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix}\begin{pmatrix} l & l' & l''' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l'''}\frac{1}{2l''+1}, l \leq l'.$$

$$E = -\frac{\lambda_1 a \lambda_2 b}{4R}\int_0^\infty \frac{dx}{x}\sum_{ll'l''}(2l+1)(2l'+1)(2l''+1) \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K^2_{l''+1/2}(x) I^2_{l+1/2}(xa/R) I^2_{l'+1/2}(xb/R).$$

# Power series expansion

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in $a/R, b/R < 1$. This expansion yields the infinite series

$$E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R} \frac{ab}{R^2} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{m=0}^{n} D_{n,m} \left(\frac{a}{R}\right)^{2(n-m)} \left(\frac{b}{R}\right)^{2m}$$

where by inspection of the first several $D_{n,m}$ coefficients we can identify them as

$$D_{n,m} = \frac{1}{2} \binom{2n+2}{2m+1},$$

# Closed form

and now we can immediately sum the expression for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln\left(\frac{1-\left(\frac{a+b}{R}\right)^2}{1-\left(\frac{a-b}{R}\right)^2}\right).$$

# PFA

Again, when $d = R - a - b \ll a, b$, the proximity force theorem is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \quad d \ll a, b.$$

However, as the figures demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is ($b/a = \mu$)

$$\frac{E}{U} = 1 + \frac{\ln[(1+\mu)^2/2\mu]}{\ln d/R}, \quad d \ll a, b.$$

# $a = b$; truncation at 100 shown

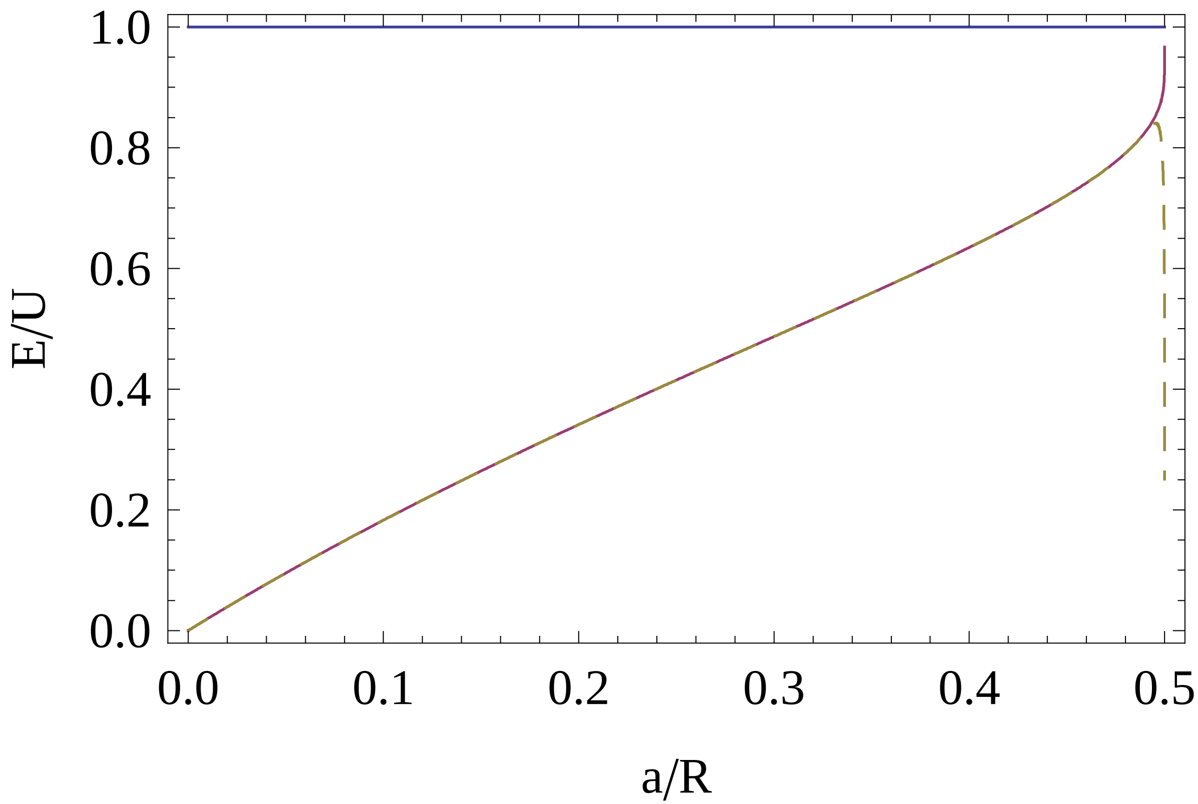



Figure 4: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

# $b/a = 49$

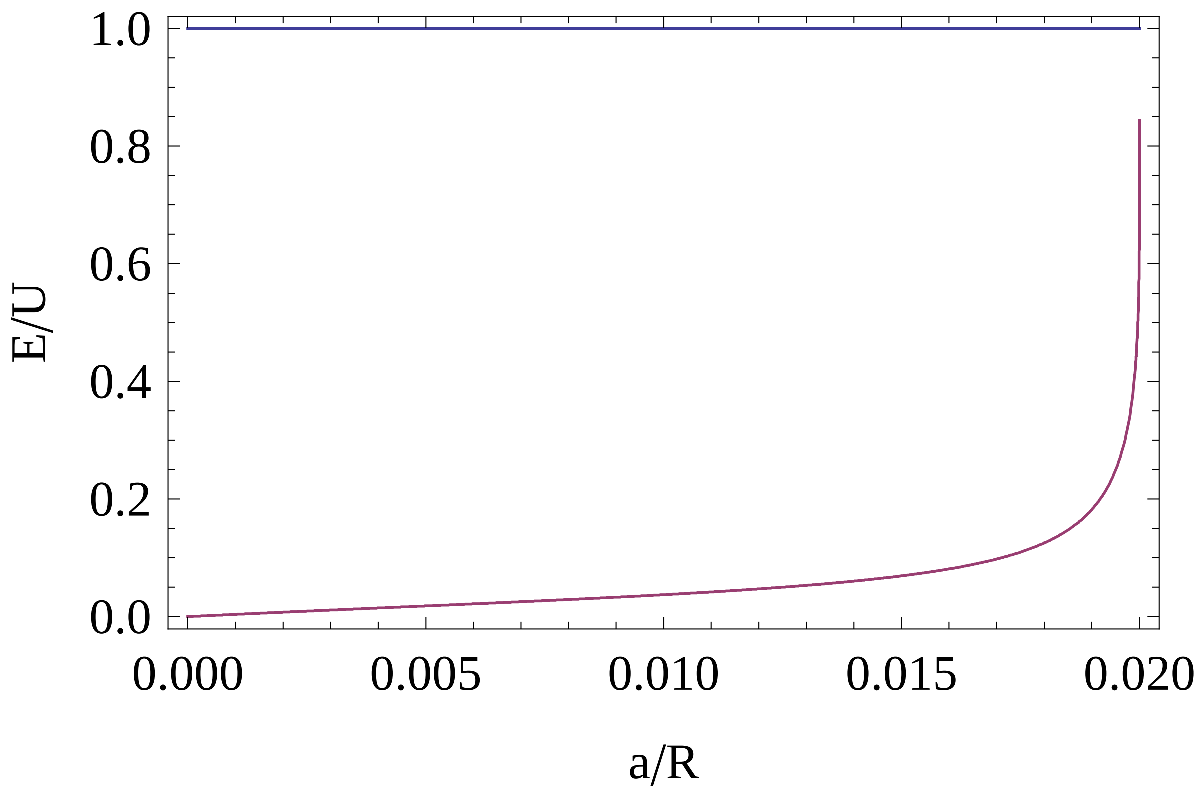



Figure 5: Plotted is the ratio of the exact interaction energy of two weakly-coupled spheres to the proximity force approximation

# Exact plane/sphere energy

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^{l}(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0},$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R}\int_0^\infty dx \sum_{l=0}^{\infty}\sqrt{\frac{\pi}{2x}}(2l+1)K_{1/2}(x)I^2_{l+1/2}\left(x\frac{a}{R}\right)$$

Then in terms of $R/2$ as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi}\left(\frac{a}{R}\right)^2 \frac{1}{1-(2a/R)^2},$$

which as $a \to R/2$ reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda}{8\pi}\frac{a}{d}.$$

# Exact energy vs. PFA

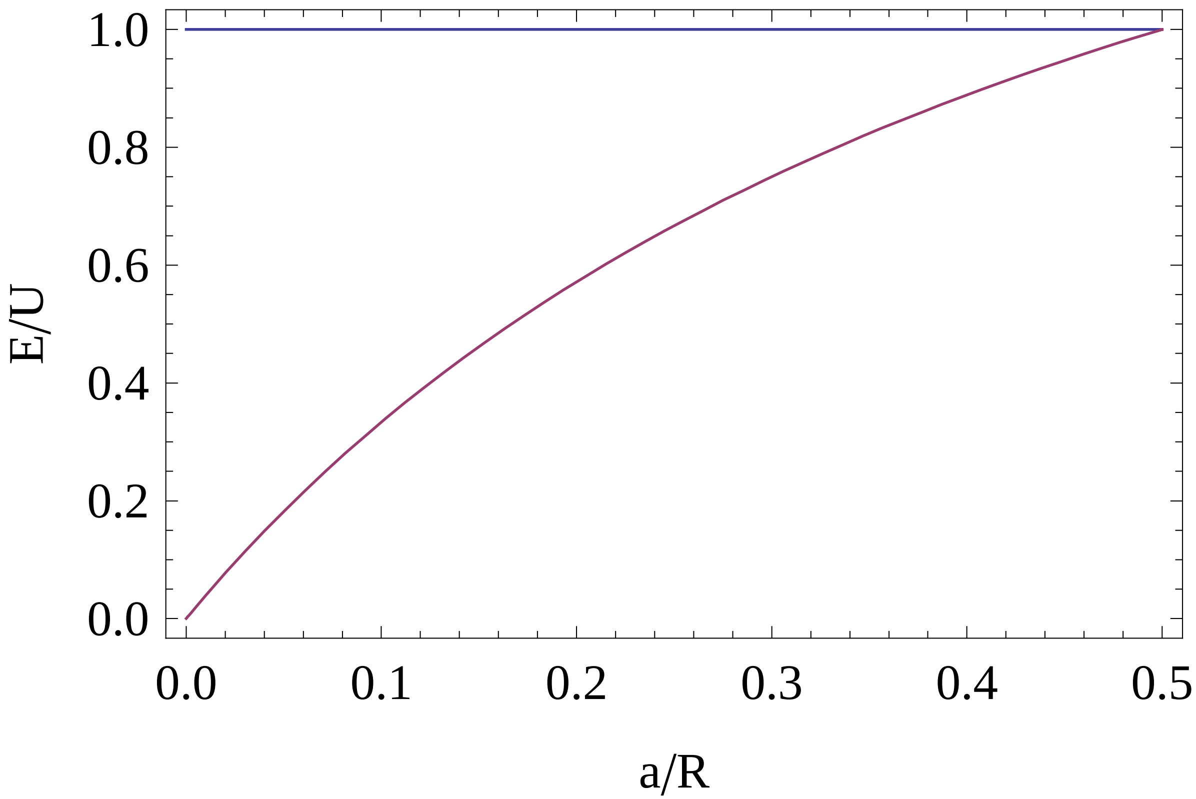



Figure 6: Plotted is the ratio of the exact interaction energy of a weakly-coupled sphere above a Dirichlet plane to the PFA.

# IV. Exact Results–Weak Coupling

In weak coupling it is possible to derive the exact (scalar) interaction between two potentials

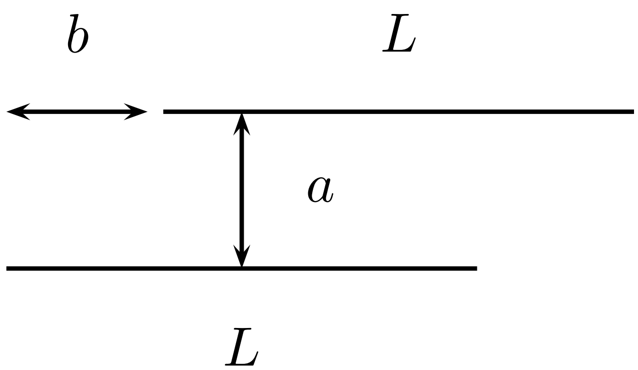
$$2D: \quad \frac{E}{L_z} = -\frac{1}{32\pi^3}\int (d\mathbf{r}_\perp)(d\mathbf{r}'_\perp)\frac{V_1(\mathbf{r}_\perp)V_2(\mathbf{r}'_\perp)}{|\mathbf{r}-\mathbf{r}'|^2},$$

$$3D: \quad E = -\frac{1}{64\pi^3}\int (d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

# Exact Results for Finite Plates

Consider two plates of finite length $L$, offset by an amount $b$, separated by a distance $a$:

$$V_1(\mathbf{r}_\perp) = \lambda_1 \delta(y)\theta(x)\theta(L-x),$$
$$V_2(\mathbf{r}'_\perp) = \lambda_2 \delta(y'-a)\theta(x'-b)\theta(L+b-x'),$$



# Exact Results for Finite Plates (cont

This gives an explicit result for the energy between the plate

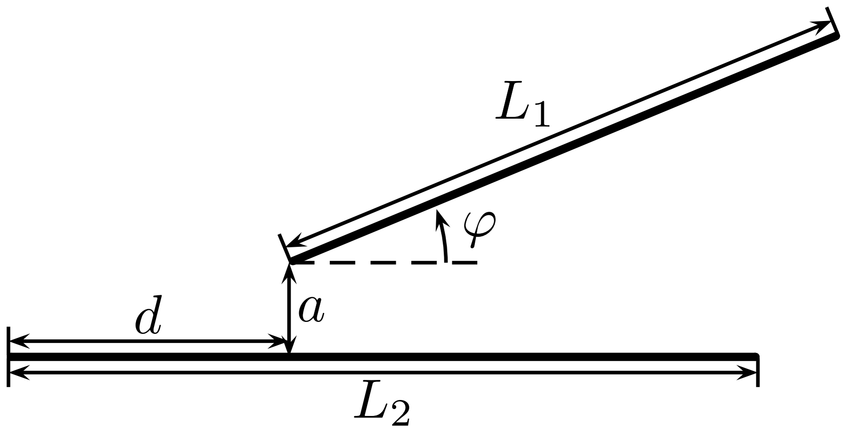
$$\frac{E}{L_z} = -\frac{\lambda_1\lambda_2}{32\pi^3}\left[-2g(b/a) + g((L-b)/a) + g((L+b)/a)\right]$$

where

$$g(x) = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) = -\text{Re}(1+ix)\ln(1+ix).$$

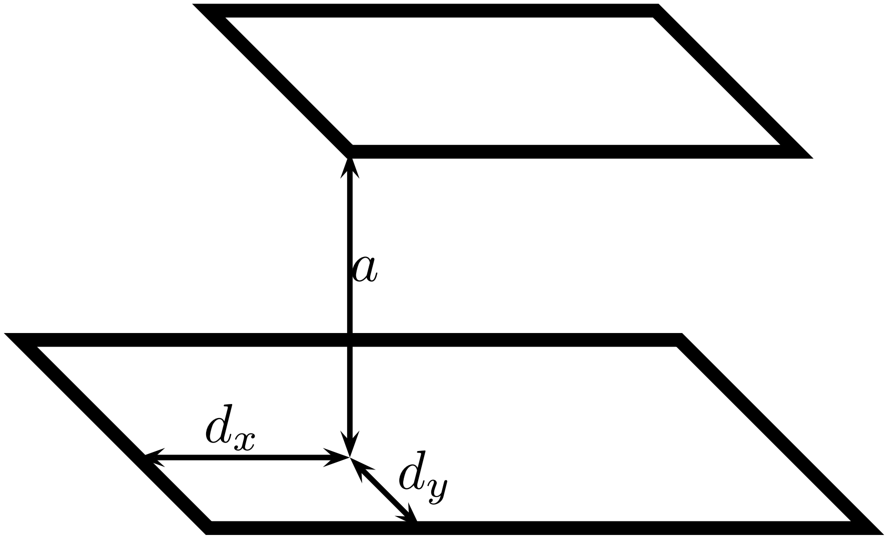
We can consider arbitrary lengths and orientations, in 3 dimensions, for the plates. [J. Wagner et al.]

# Tilted plates




Explicit interaction energy can be given in terms of $\mathrm{Ti}_2$, inverse tangent integral. For fixed CM, for $L_1 \to L$, $L_2 \to \infty$, $d \to -\infty$, and $L > 2a$, equilibrium position is at $\phi = \pi/2$.

# Rectangular Parallel Plates




As $a \longrightarrow 0$,

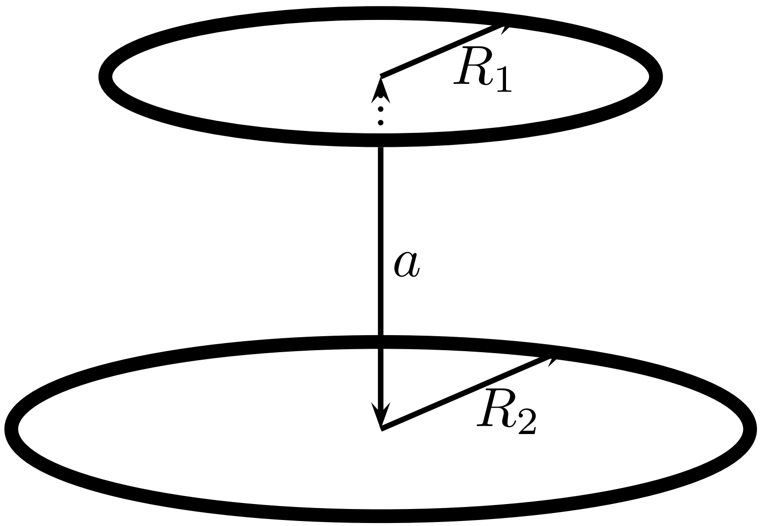
$$\frac{F}{A} = -\frac{\lambda_1 \lambda_2}{32\pi^2 a^2}(1 + c_1 a + c_2 a^2 + \ldots)$$

# Correction to Lifshitz formula

- If upper plate is completely above lower plate, $c_1 = 0$.
- If plates are of the same size and aligned,

$$c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}.$$

# Coaxial disks




- If $R_1 < R_2$, $c_1 = 0$.
- If $R_1 = R_2$, $c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}$.

# Salient Features–two thin plates

- Two plates of finite length experience a lateral force so that they wish to align in the position of maximum symmetry.
- In this symmetric configuration, there is a torque about the CM of a single plate so that it tends to seek perpendicular orientation with respect to the other plate.
- First correction to Lifshitz formula is geometrical.

# Conclusions to Lecture I

- We have illustrated the multiple scattering or $TGTG$ formalism for scalar fields.
- We have given examples for semitransparent cylinders, spheres, and planes, which include weak and strong coupling.
- In the next lecture, we shall consider electromagnetic interactions.