# A Derivative Expansion for the Casimir Effect

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*In collaboration with:*

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10-10-2012

# Introduction

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- *Based on a time-honored idea by Derjaguin:*
  - B.V. Derjaguin, Koll. Z. 69, N2, 155 (1934);
  - B.V. Derjaguin and I. I. Abrikosova, Sov. Phys. JETP **3**, 819 (1957);
  - B.V. Derjaguin, Sci. Am. **203**, 47 (1960).

# Sketch of the PFA

Two surfaces, one flat ($L$), the other ($R$) describable by a single Monge patch:

$$\begin{aligned} L)\ x_3 &= 0 \\ R)\ x_3 &= \psi(x_1, x_2) \end{aligned}$$

# Sketch of the PFA

Two surfaces, one flat ($L$), the other ($R$) describable by a single Monge patch:

$$L)\ x_3 = 0$$
$$R)\ x_3 = \psi(x_1, x_2)$$

The PFA energy is:

$$E_{PFA} = \int d^2\mathbf{x}_\parallel \, \mathcal{E}_{pp}[\psi(\mathbf{x}_\parallel)]$$

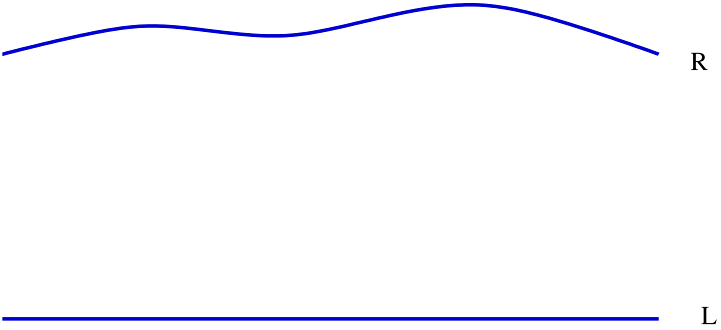
$\mathcal{E}_{\rm pp}(a)$: energy per unit area for two parallel plates.

For a massless real scalar field and two (infinite) parallel Dirichlet surfaces at a distance $a$:

$$\mathcal{E}_{\rm pp}(a) = -\frac{\pi^2}{1440\, a^3},$$

# PFA

Around each point on the surface, one approximates the surface element by a constant-$x_3$ plane:



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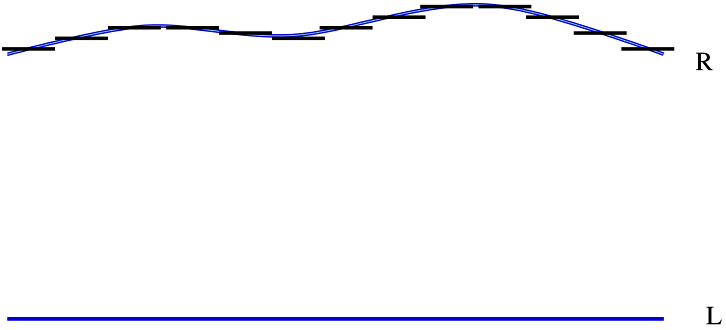


Each plane contributes: $d^2\mathbf{x}_{||}\ \mathcal{E}_{pp}[\psi(\mathbf{x}_{||})]$

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- In particular, we would like to be able to assess the (in)accuracy of the PFA, without having to solve the exact problem.
- In other words, going beyond the PFA at the lowest possible cost.

# Derivative expansion (DE) approach

I will talk about results presented in:

F.D. Lombardo, F.D.Mazzitelli and C. D. F.; Phys.Rev. D84 (2011) 105031;
Phys. Rev. D86 (2012) 045021;
Phys. Rev. D85 (2012) 125037

Interesting generalizations and extensions in:

G. Bimonte, T. Emig, R. L. Jaffe, M. Kardar. Europhys.Lett.,97, 50001 (2012);
G. Bimonte, T. Emig, M. Kardar. Appl.Phys.Lett.,100,074110 (2012).

## Motivation for a DE (single Monge patch)

- The Casimir energy $E$ is a **functional** of the *shape* (i.e. $\psi$) of the surfaces. PFA should be the form adopted by the functional for *almost flat* surfaces.

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- Thus, the expansion we look for should reproduce, to lowest order, the PFA.
- Terms involving **derivatives** of $\psi$ should appear in the corrections to the PFA.

# Idea of the DE

In QFT, there is an approach to evaluating the **effective action** that does exactly that:

- C. M. Fraser, Z. Phys. C 28, 101 (1985); I. J. R. Aitchison and C. M. Fraser, Phys. Lett. 146B, 63 (1984); ibid. Phys. Rev. D 31, 2605 (1985).
- R. Jackiw, Phys. Rev. D 9, 1686 (1974).
- L. H. Chan, Phys. Rev. Lett. 54, 1222 (1985); ibid. 55, 21 (1985).

**Basic idea**

To perform an expansion of the effective action (due to an integrated fluctuating field) according to the number of derivatives of its arguments (the 'classical' field).

In the Casimir context: classical field $= \psi$.

# Vacuum energy from the effective action

Total vacuum energy $E$:

$$E = \lim_{T\to\infty} \left[ \frac{1}{T} \Gamma(\psi) \right]$$

where:

$$\Gamma = -\log \frac{\mathcal{Z}[\psi]}{\mathcal{Z}_0}$$

$T$: length of the (imaginary) time interval;
$\mathcal{Z}[\psi]$: vacuum amplitude in the presence of the mirrors;
$\mathcal{Z}_0$: free (no surfaces) vacuum amplitude;
($E$: also the $\beta \to \infty$ limit of the free energy);
We shall subtract self-energies.
$\psi$: time-independent (this may be imposed at the end of the calculation).

# Form of the DE for Γ (2nd order)

Up to order two in derivatives of $\psi$:

$$\Gamma[\psi] \;=\; \int d^3x_\parallel \left[V_{eff}(\psi) \,+\, z(\psi)(\partial\psi)^2\right] .$$

$(x_\parallel \equiv (x_0, x_1, x_2))$

## Constraints

1) Dimensionful quantities: $\psi$ and its derivatives; 2) It should reduce to the PFA for sufficiently smooth surfaces; 3) Locality.

Then:

$$\Gamma[\psi] \;=\; \int d^3x_\parallel \, \frac{1}{\psi(x_\parallel)^3}\left[b_0 \,+\, b_2\,(\partial\psi)^2\right] .$$

$b_0$ and $b_2$: dimensionless numbers that completely determine the approximation. Note the 'universality' of the coefficients, namely, they are the same, regardless of the (smooth) surface considered.

# 'Constructive' approach

The previous equation may also be obtained in a more constructive way. If one knows $\Gamma(\psi)$ for:

- $\psi(x_\parallel) = a + \eta(x_\parallel)$, $|\eta| << a$, to second order in $\eta$.
- Second order in derivatives of $\eta$, such that:

$$\Gamma(a+\eta) \simeq \int d^3x_\parallel \left[ V_{eff}(a) + z(a)(\partial\psi)^2 \right] ,$$

then, to second order in derivatives of $\psi$ (and no assumption about magnitude of $\eta$):

$$\Gamma[\psi] = \int d^3x_\parallel \left[ V_{eff}(\psi) + z(\psi)(\partial\psi)^2 \right] .$$

Namely: replace *a* by $\psi$ in the *a*-dependent coefficients of the expansion.
The proof relies on the resummation of all the orders in $\eta$, in the expansion of $\Gamma$ to second order in derivatives.

# Determination of the $b_0$, $b_2$ coefficients

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- More general approach: to determine them from the expansion of $\Gamma$ for:

$$\psi(x_{||}) = a + \eta(x_{||})$$

  with $|\eta| << a$ and smooth, to second order in derivatives of $\eta$.
- Then $b_0$ and $b_2$ may be determined from the knowledge of $E$ up to the second order in $\eta$ and second order in derivatives. In particular, $b_0 = -\frac{\pi^2}{1440}$ is derived from the pp result (real scalar field).

# Effective action (Dirichlet)

We need the effective action for a scalar field with Dirichlet Boundary Conditions.
Based on the functional representation for the CE:

H. Li and M. Kardar, PRA **46**, 6490 (1992); T. Emig, A. Hanke, R. Golestanian, and M. Kardar, PRL **87**, 260402 (2001); ibidem Phys. Rev. **A67**, 022114 (2003).

Functional representation of $\mathcal{Z}(\psi)$:

$$\mathcal{Z}(\psi) \;=\; \int \mathcal{D}\varphi \; \delta_L(\varphi)\delta_R(\varphi) \; e^{-\mathcal{S}_0(\varphi)} \;,$$

$\delta_A(\varphi)$, $A = L, R$: functional $\delta$ function which imposes $D$ conditions on the respective mirror, $\mathcal{S}_0$: free Euclidean action:
$\mathcal{S}_0 = \frac{1}{2} \int dx_0 \int d^3x \, (\partial\varphi)^2$.

# Auxiliary fields

Exponentiate $\delta$-functionals with two auxiliary fields, $\lambda_L$ and $\lambda_R$, functions of $x_\parallel \equiv (x_0, x_1, x_2) \equiv (x_0, \mathbf{x}_\parallel)$, also satisfying periodic boundary conditions in the $x_0$ coordinate.
Dirichlet case:

$$\begin{aligned} \delta_L(\varphi) &= \int \mathcal{D}\lambda_L \; e^{i \int d^3x_\parallel \; \lambda_L(x_\parallel)\, \varphi(x_\parallel, 0)} \\ \delta_R(\varphi) &= \int \mathcal{D}\lambda_R \; e^{i \int d^3x_\parallel \; \sqrt{g(x_\parallel)}\, \lambda_R(x_\parallel)\, \varphi(x_\parallel, \psi(x_\parallel))} \;, \end{aligned}$$

$g$ is the determinant of $g_{\alpha\beta}$, the induced metric on $R$:

$$g_{\alpha\beta}(x_\parallel) = \delta_{\alpha\beta} + \partial_\alpha \psi(x_\parallel) \partial_\beta \psi(x_\parallel) \;,$$

# Derivation

Using the exponential representations above and integrating over $\varphi$,

$$\mathcal{Z}(\psi) \;=\; \mathcal{Z}^{(0)} \int \mathcal{D}\lambda_L \mathcal{D}\lambda_R \; e^{-\frac{1}{2}\int_{x_\parallel, x'_\parallel} \lambda_A(x_\parallel)\mathbb{T}_{AB}(x_\parallel, x'_\parallel)\lambda_B(x'_\parallel)} ,$$

where a self-energy contribution has been discarded.

# The $\mathbb{T}$ Dirichlet kernel matrix

$$T_{LL}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | x'_{\parallel}, 0 \rangle \quad (1)$$

$$T_{LR}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, 0 | (-\partial^2)^{-1} | x'_{\parallel}, \psi(x'_{\parallel}) \rangle \quad (2)$$

$$T_{RL}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^2)^{-1} | x'_{\parallel}, 0 \rangle \quad (3)$$

$$T_{RR}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel}, \psi(x_{\parallel}) | (-\partial^2)^{-1} | x'_{\parallel}, \psi(x'_{\parallel}) \rangle \quad (4)$$

$$\langle x | (-\partial^2)^{-1} | y \rangle = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2} \equiv \Delta(x-y) \, .$$

# 2nd order expansion

Now, split $\psi$ into two components,

$$\psi(\mathbf{x}_{\parallel}) \;=\; a + \eta(\mathbf{x}_{\parallel}) \;.$$

$a$: spatial average of $\psi$ and $\eta$ contains its varying piece.
Expand $\Gamma$ up to the second order in $\eta$, keeping up to the second order term in an expansion in derivatives.
Expanding the matrix $\mathbb{T}$ in powers of $\eta$

$$\mathbb{T} = \mathbb{T}^{(0)} + \mathbb{T}^{(1)} + \mathbb{T}^{(2)} + \ldots \,,$$

we obtain $\Gamma \;=\; \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} \,+\, \ldots$, where

$$\Gamma^{(0)} = \frac{1}{2}\mathrm{Tr}\log\mathbb{T}^{(0)} \;\;, \qquad \Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}\right] \;,$$

$$\Gamma^{(2)} = \frac{1}{2}\mathrm{Tr}\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(2)}\right] - \frac{1}{4}\mathrm{Tr}\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}\right]$$

# Zeroth-order term (PFA)

Zeroth-order term may be obtained by replacing first $\psi$ by a constant, $a$, and then subtracting the contribution corresponding to $a \to \infty$. This yields,

$$\Gamma^{(0)} = \frac{1}{2} \mathrm{Tr} \log \left[ 1 - (T_{LL}^{(0)})^{-1} T_{LR}^{(0)} (T_{RR}^{(0)})^{-1} T_{RL}^{(0)} \right]$$

$T_{\alpha\beta}^{(0)}$: identical to the ones for two flat parallel mirrors at a distance $a$ apart. After evaluating the trace:

$$V_{eff}(\psi) = \frac{1}{2} \int \frac{d^3 k_\parallel}{(2\pi)^3} \log \left[ 1 - e^{-2k_\parallel \psi(x_\parallel)} \right],$$

which yields the PFA aproximation to the vacuum energy.

# Second order term

(First order term vanishes identically.) For the second order one we have two contributions:

$$\Gamma^{(2)} = \Gamma^{(2,1)} + \Gamma^{(2,2)}$$

$$\Gamma^{(2,1)} = \frac{1}{2}\mathrm{Tr}\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(2)}\right]$$

and

$$\Gamma^{(2,2)} = -\frac{1}{4}\mathrm{Tr}\left[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}\right] ,$$

where we have to keep up to two derivatives of $\eta$.

In Fourier space, and before expanding to second order in momentum (derivatives):

$$\Gamma^{(2,j)} = \frac{T}{2} \int \frac{d^2k}{(2\pi)^2} f^{(2,j)}(\mathbf{k}) \, |\tilde{\eta}(\mathbf{k})|^2$$

$(j = 1, 2)$, $\mathbf{k} = (k_1, k_2)$, $f^{(2,j)}$ are the kernels:

$$\begin{aligned} f^{(2,1)}(k) &= -\int \frac{d^3p}{(2\pi)^3} \frac{|p| \, |p+k|}{1 - e^{-2|p+k|a}} \\ f^{(2,2)}(k) &= -\int \frac{d^3p}{(2\pi)^3} \frac{|p||p+k| e^{-2|p+k|a}(1 + e^{-2|p|a})}{(1 - e^{-2|p|a})(1 - e^{-2|p+k|a})} . \end{aligned}$$

We need to subtract an *a*-independent (self-energy) contribution. Putting together the results from the two terms above:

$$\Gamma^{(2)} \;=\; \frac{T}{2} \int \frac{d^2k}{(2\pi)^2} f^{(2)}(\mathbf{k}) \, |\tilde{\eta}(\mathbf{k})|^2$$

with:

$$f^{(2)}(k) \;=\; -2 \int \frac{d^3p}{(2\pi)^3} \frac{|p| \, |p+k|}{(1-e^{-2|p|a})(e^{2|p+k|a}-1)}$$

where we just need to extract its $\mathbf{k}^2$ term in a Taylor expansion at zero momentum. Namely $f^{(2)}(\mathbf{k}) \;\simeq\; \chi \, \mathbf{k}^2$, where

$$\begin{aligned} \chi \;&=\; \frac{1}{2} \Big[ \frac{\partial^2 f^{(2)}(k)}{\partial k^2} \Big]_{k \to 0} \\ &=\; -\int \frac{d^3p}{(2\pi)^3} \frac{|p|}{(1-e^{-2|p|a})} \lim_{k\to 0} \frac{\partial^2}{\partial k^2} \Big[ \frac{|p+k|}{(e^{2|p+k|a}-1)} \Big]. \end{aligned}$$

The resulting integral may be exactly calculated,

$$\chi = -\frac{\pi^2}{1080\, a^3} .$$

Thus,

$$\begin{aligned} \Gamma^{(2)}(a,\eta) &= -\frac{T}{2}\frac{\pi^2}{1080}\int \frac{d^2k}{(2\pi)^2}\frac{\mathbf{k}^2}{a^3}\,|\tilde{\eta}(\mathbf{k})|^2 \\ &= -\frac{T}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel\,\frac{1}{a^3}\,(\partial_\alpha\eta)^2 , \end{aligned}$$

where, to obtain the second order contribution in derivatives to the vacuum energy, we need to replace $a \to \psi$, $\eta \to \psi$, and cancel the $T$ factor, obtaining:

$$E^{(2)}_{\rm vac} = \frac{\Gamma^{(2)}(\psi)}{T} = -\frac{1}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel\,\frac{(\partial_\alpha\psi)^2}{\psi^3} ,$$

($\alpha$ runs from 1 to 2).

# The final result

Putting together the terms, the DE to second order yields:

$$E_{DE} \equiv = -\frac{\pi^2}{1440} \int d^2\mathbf{x}_\parallel \, \frac{1}{\psi^3} \left[ 1 + \frac{2}{3} (\partial_\alpha \psi)^2 \right] .$$

- Remarkably simple formula.
- First term: PFA for the Casimir energy.
- Second term: contains first non-trivial correction to the PFA.
- A straightforward calculation provides the relative weight between both terms.
- It is local.
- Amounts to using a better local approximation to the energy density for two surface elements.
- Other (perfect) boundary conditions: different constant coefficients.

# Sinusoidal corrugations in $x_1$ direction

$$\psi(x_1) = a + \epsilon \sin \frac{2\pi x_1}{\lambda} \,.$$

$$E_{DE} = \frac{\pi^2}{1440} \left[ \int d^2\mathbf{x}_\parallel \, \frac{1}{(a + \epsilon \sin \frac{2\pi x_1}{\lambda})^3} \left( 1 + \frac{2}{3} \left( \frac{2\pi}{\lambda} \right)^2 \epsilon^2 \cos^2 \frac{2\pi x_1}{\lambda} \right) \right]$$

In this case, the derivative expansion should be reliable when $\lambda$ is much larger than both $\epsilon$ and $a$. Should therefore be the largest relevant distance in the problem.

Expanding for $\epsilon \ll a$,

$$E_{DE} = \frac{\pi^2 L^2}{1440 a^3} \left[ 1 + 3 \left( \frac{\epsilon}{a} \right)^2 + \frac{4\pi^2}{3} \left( \frac{\epsilon}{\lambda} \right)^2 \right] \,.$$

in perfect agreement with the expansion of the result by Emig et al, when expanded for $a << \lambda$.

# A sphere in front of a plane

Sphere of radius $R$ at a distance $d$ from a plane.
Expect the derivative expansion to work well in the large-$R$ limit.

$$\psi(\rho) = a + R(1 - \sqrt{1 - \frac{\rho^2}{R^2}}) .$$

(hemisphere, $0 \leq \rho \leq R$). DE well defined if we restrict the integrations to the region $0 \leq \rho \leq \rho_M < R$.
The DE integral can be evaluated. Results in a rather long analytic expression $E_{DE}(\rho_M, a, R)$.
Expanded in inverse powers of $R^{-1}$:

$$E_{DE} \simeq \frac{\pi^3}{1440} \frac{R}{a^2} \left( 1 + \frac{1}{3} \frac{a}{R} \right) .$$

Up to this order, the result does not depend on $\rho_M$. In agreement with the asymptotic expansion obtained from the exact formula for this configuration

# Cylinder in front of a plane

Cylinder of radius $R$ at a distance $a$ from a plane. Function $\psi$ given by:

$$\psi(x_1) = a + R(1 - \sqrt{1 - \frac{x_1^2}{R^2}}),$$

with $-x_M < x_1 < x_M < R$ (to cover the part of the cylinder which is closer to the plane). Result:

$$E_{DE} \simeq \frac{\pi^3 L}{1920\sqrt{2}} \frac{R^{1/2}}{a^{5/2}} \left(1 + \frac{7}{36}\frac{a}{R}\right).$$

The result does not depend on $x_M$. In agreement with the asymptotic expansion obtained from the exact formula.

# Higher dimensions and thermal effects

F.D.L, F.D.M. and C.D.F. Phys. Rev. D86 (2012) 045021

Is the DE always well-defined?
What we have found ($d$ spatial dimensions):

- Real scalar field with either $D$ or $N$ boundary conditions on both mirrors (one flat, one gently curved).
- $D$ boundary conditions: NTLO term is quadratic in derivatives ($\forall d$).
- $N$ boundary conditions: the same holds true for $d \neq 2$. $d = 2$, NTLO term becomes nonlocal. Same nonlocality for $d = 3$ ('dimensional reduction') at high temperatures.
- $N$ mirrors in $d = 3$, we find a nonlocal NTLO term for any $T > 0$.

# The system

Free energy: $\Gamma_\beta(\psi)$,
Vacuum energy: $E_{\rm vac}(\psi) = \lim_{\beta\to\infty} \Gamma_\beta(\psi)$.

$$\Gamma_\beta(\psi) \,=\, -\frac{1}{\beta}\,\log\big[\frac{\mathcal{Z}_\beta(\psi)}{\mathcal{Z}_\beta^{(0)}}\big]\;.$$

# Derivative expansion for $D$

Same idea and approach, but different number of spatial dimensions $d$. Time coordinate is compact (periodic).
Keeping up to two derivatives, the Casimir free energy can be written as follows:

$$\Gamma_\beta(\psi) \;=\; \int d^{d-1}\mathbf{x}_\parallel \left\{ b_0(\frac{\psi}{\beta}) \frac{1}{[\psi(\mathbf{x}_\parallel)]^d} \;+\; b_2(\frac{\psi}{\beta}) \, \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_\parallel)]^d} \right\}$$

Dimensionless functions $b_0$ and $b_2$ can be obtained from the knowledge of the Casimir free energy for small departures around the $\psi(\mathbf{x}_\parallel) = a = \mathrm{constant}$ case.

$$
\begin{aligned}
b_0(\xi) &= \frac{\xi}{2} \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1}\mathbf{p}_\parallel}{(2\pi)^{d-1}} \log\left[1 - e^{-2\sqrt{(2\pi n\xi)^2 + \mathbf{p}_\parallel^2}}\right] \\
&= \xi \frac{\pi^{(1-d)/2}}{2^{d-1}\Gamma(\frac{d-1}{2})} \sum_{n=-\infty}^{+\infty} \int_0^\infty d\rho \rho^{d-2} \log\left[1 - e^{-2\sqrt{(2\pi n\xi)^2 + \rho^2}}\right]
\end{aligned}
$$

$$
b_2(\xi) = \frac{1}{2}\left[\frac{\partial F^{(2)}(\xi; n, |\mathbf{l}_\parallel|)}{\partial |\mathbf{l}_\parallel|^2}\right]_{n\to 0, |\mathbf{l}_\parallel|\to 0} .
$$

where

$$
F^{(2)}(\xi; n, |\mathbf{l}_\parallel|) = -2\xi \sum_{m=-\infty}^{+\infty} \int \frac{d^{d-1}\mathbf{p}_\parallel}{(2\pi)^{d-1}} \Big\{
$$

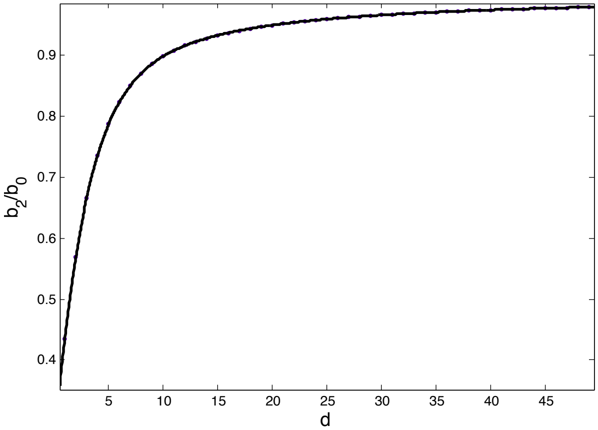
$$
\times \sqrt{(2\pi m\xi)^2 + \mathbf{p}_\parallel^2} \sqrt{(2\pi(m+n)\xi)^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2}
$$

$$
\times \frac{1}{1 - \exp\left[-2\sqrt{(2\pi m\xi)^2 + \mathbf{p}_\parallel^2}\right]} \frac{1}{\exp\left\{2\sqrt{[2\pi(m+n)\xi]^2 + (\mathbf{p}_\parallel + \mathbf{l}_\parallel)^2}\right\} - 1} \Big\}
$$

# Low and high temperature limits

Limits of the two coefficients $b_0$ and $b_2$; determine the form of the DE in the corresponding limits.
Relevant scale to compare the temperature with: inverse of the distance between the mirrors. Zero temperature limit: $\xi \to 0$

| | $\frac{b_2(d)}{b_0(d)}$ | $\sim$ |
|---|---|---|
| $d = 1$ | $\frac{1}{\pi^2}(1 + \frac{\pi^2}{3})$ | 0.435 |
| $d = 2$ | $\frac{1+6\zeta(3)}{12\zeta(3)}$ | 0.569 |
| $d = 3$ | $2/3$ | 0.667 |
| $d = 4$ | $\frac{-\zeta(3)+10\zeta(5)}{12\zeta(5)}$ | 0.737 |
| $d = 5$ | $\frac{10\pi^2-21}{10\pi^2}$ | 0.787 |
| $d = 6$ | $\frac{-2\zeta(5)+7\zeta(7)}{6\zeta(7)}$ | 0.824 |


0.4
0.5
0.6
0.7
0.8
0.9
$b_2/b_0$
5
10
15
20
25
30
35
40
45
d

# High temperature limit

$\xi >> 1$

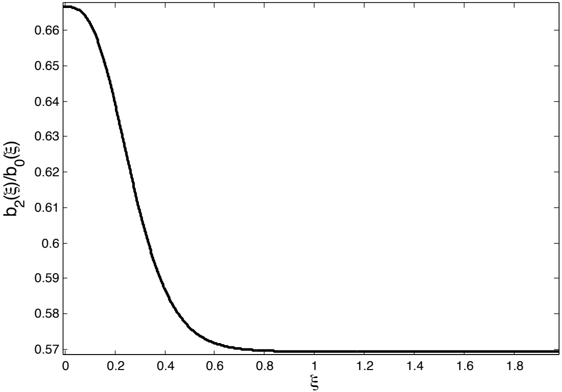
$$\left[\Gamma_\beta(\psi, d)\right]_{\psi/\beta>>1} \sim$$

$$\sim \frac{1}{\beta} \int d^{d-1}\mathbf{x}_\parallel \left\{ b_0(d-1) \frac{1}{[\psi(\mathbf{x}_\parallel)]^{d-1}} + b_2(d-1) \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_\parallel)]^{d-1}} \right\}.$$

('dimensional reduction' phenomenon at high temperatures).
In particular, for $d = 3$:

$$\left[\Gamma_\beta(\psi, 3)\right]_{\psi/\beta>>1} \sim -\frac{\zeta(3)}{16\pi\beta} \int d^2\mathbf{x}_\parallel \frac{1}{[\psi(\mathbf{x}_\parallel)]^2} \left\{1 + 0.569\,(\partial\psi)^2\right\} .$$

# Ratio between $b_2(\xi)$ and $b_0(\xi)$ as a function of $\xi$ $(d = 3)$.



# Neumann conditions

Now:

$$\Gamma_\beta(\psi) \;=\; \int d^{d-1}\mathbf{x}_\parallel \left\{ c_0(\frac{\psi}{\beta}) \frac{1}{[\psi(\mathbf{x}_\parallel)]^d} \;+\; c_2(\frac{\psi}{\beta}) \, \frac{(\partial\psi)^2}{[\psi(\mathbf{x}_\parallel)]^d} \right\}$$

with two (new) dimensionless functions $c_0$ and $c_2$.

- An explicit calculation shows that the NTLO term is local and quadratic in derivatives for $d \neq 2$.
- $d = 2$ at zero temperature, or when $d = 3$ and the temperature is not zero: NTLO term is nonlocal.

# The coefficients

The zero order term coincides with the one for the Dirichlet case; namely: $c_0 = b_0$.

| | $\frac{c_2(d)}{c_0(d)}$ | $\sim$ |
|---|---|---|
| $d = 1$ | $\frac{1}{3} - \frac{\zeta(0)}{3\zeta(2)}$ | $0.435$ |
| $d = 3$ | $\frac{2}{3} - \frac{4\zeta(2)}{3\zeta(4)}$ | $-1.360$ |
| $d = 4$ | $\frac{5}{6} - \frac{19\zeta(3)}{12\zeta(5)}$ | $-1.002$ |
| $d = 5$ | $1 - \frac{9\zeta(4)}{5\zeta(6)}$ | $-0.915$ |
| $d = 6$ | $\frac{7}{6} - \frac{2\zeta(5)}{\zeta(7)}$ | $-0.890$ |
| $d = 7$ | $\frac{4}{3} - \frac{46\zeta(6)}{\zeta(8)}$ | $-0.886$ |

# Neumann conditions, $d = 2$

$d = 2$: not possible to compute the NTLO term coefficient by introducing the derivative with respect to $|l_{\parallel}|^2$ inside the integral $\Rightarrow$ signal of branch cut at zero momentum. Also at $d = 3$ with $T > 0$ (thermal zero mode).

For small departures of the plane-plane geometry, up to second order in $\eta$,

$$\Gamma^{(2)}_{\infty} = \frac{1}{2} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} \left[ g^{(2)}(k_{\parallel}) \right]_{k_0 \to 0} \left| \tilde{\eta}(\mathbf{k}_{\parallel}) \right|^2 ,$$

with:

$$g^{(2)}(k_{\parallel}) = -2 \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \frac{\left[ p_{\parallel} \cdot (p_{\parallel} + k_{\parallel}) \right]^2}{|p_{\parallel}| \, |p_{\parallel} + k_{\parallel}|} \frac{1}{1 - e^{-2a|p_{\parallel}|}} \times \frac{1}{e^{2a|p_{\parallel} + k_{\parallel}|} - 1} .$$

The form factor $g^{(2)}(k_{\parallel})$ does not admit an expansion in powers of $k_{\parallel}^2$, for $d = 2$ (IR logarithmic divergence at $p_{\parallel} = 0$).

# Nonlocal NTLO term

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Nonlocality related to the existence of massless modes in the theory. These modes are generally allowed by Neumann but not for Dirichlet boundary conditions (mass gap of order $1/a$).
Similar non-analiticity for $d = 3$ at finite temperature.

# Higher orders

Next to NTLO terms:

$$|\partial\psi|^4,\ \psi|\partial\psi|^2\partial^2\psi,\ \psi^2\partial^2\psi\partial^2\psi$$

$$\psi^2\partial_\alpha\partial_\beta\psi\partial_\alpha\partial_\beta\psi,\ \psi^3\partial^2\partial^2\psi$$

(and terms more derivatives for higher orders).
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(and terms more derivatives for higher orders).
¿Can the free energy be expanded in local terms up to any desired order? Necessary condition: the expansion must hold true for a particular case: when $\psi = a + \eta$, with $\eta \ll 1$, and one keeps just the quadratic term in $\eta$.
$T = 0$, $d = 3$ dimensions. The quadratic contribution to the Casimir energy:

$$\Gamma_\infty^{(2)} = \frac{1}{2a^5} \int \frac{d^2\mathbf{k}_\parallel}{(2\pi)^2}\, h^{(2)}(0, a\mathbf{k}_\parallel) \left|\widetilde{\eta}(\mathbf{k}_\parallel)\right|^2 ,$$

where $h^{(2)}$ depends on the boundary conditions.

- On general grounds, we expect the form factor to be a function of $a(\omega^2 + \mathbf{k}_\parallel^2)^{1/2} \equiv a|k_\parallel|$, and explicit calculations confirm this fact.
- The Casimir energy will not admit an expansion in derivatives if the form factor includes odd powers or logs of its argument.
- At higher orders, and relaxing the condition of a quadratic approximation in $\eta$ new non analytic terms may arise, which can be of the same order in the DE as the ones that come from the term quadratic in $\eta$.

# Electromagnetic field

Extension of the improved PFA to the case of two imperfect thin mirrors.

F. Lombardo, F.D. Mazzitelli and C.D. F., Phys. Rev. D85 (2012) 125037

The action $\mathcal{S}$ for this model is assumed to have the structure:

$$\begin{aligned} \mathcal{S} &= \mathcal{S}(A; y_L, y_R) \\ &= \mathcal{S}_0(A) + \mathcal{S}_L(A; y_L) + \mathcal{S}_R(A; y_R) , \end{aligned}$$

where $A$ denotes the 4-potential and $\mathcal{S}_0$ its free action:

$$S_0(A) = \int d^4x \left[ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{b}{2} (\partial_\mu A_\mu)^2 \right] ,$$

$\mathcal{S}_L$ and $\mathcal{S}_R$ are terms that couple $A$ to each mirror, with $y_L$ and $y_R$ denoting parametrizations of their respective surfaces.
Assumming that $\mathcal{S}_\Sigma$ is quadratic in $A_\mu$:

$$\mathcal{S}_\Sigma(A; y) = \frac{1}{4} \int d^3\sigma \sqrt{g(\sigma)}\, d^3\sigma' \sqrt{g(\sigma')}\, F_{\alpha\beta}(\sigma) F_{\alpha'\beta'}(\sigma') \pi^{\alpha\beta\alpha'\beta'}(\sigma, \sigma') ,$$

# The model and its PFA

We shall consider a particular class of models, corresponding to:

$$\mathcal{S}_\Sigma(A;y) \;=\; \frac{\lambda}{4}\int d^3\sigma\,\sqrt{g(\sigma)}\,F_{\alpha\beta}F^{\alpha\beta}\;,$$

where $\lambda$ is a constant (can be extended to include frequency dependent couplings).
The improved PFA reads:

$$E_{\rm vac} \;\simeq\; E^{(0)}_{\rm vac} + E^{(2)}_{\rm vac} + \ldots$$

where the index denotes the order in derivatives.

$$E^{(0)}_{\rm vac} \;=\; \int d^2\mathbf{x}_\parallel \int \frac{d^3k}{(2\pi)^3}\,\ln\Big[1 - \frac{(\frac{|k|}{2})^2}{(\frac{1}{\lambda_L}+\frac{|k|}{2})(\frac{1}{\lambda_R}+\frac{|k|}{2})}\,e^{-2|k|\psi(\mathbf{x}_\parallel)}\Big]\;.$$

# NTLO term, imperfect mirrors

The NTLO correction to the PFA has the form

$$E^{(2)}_{\rm vac} = \frac{1}{2}\int d^2\mathbf{x}_\parallel c(\psi)(\partial_j\psi)^2 \,.$$

Let us first consider the perfect-mirror limit. One can derive the idealized limit of perfect conductivity by taking $\lambda_1, \lambda_2 \to \infty$. The leading term reads, in this case

$$\left[E^{(0)}_{\rm vac}\right]_{\rm perf} = \int d^2\mathbf{x}_\parallel \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - e^{-2|k|\psi(\mathbf{x}_\parallel)}\right] = -\frac{\pi^2}{720}\int \frac{d^2\mathbf{x}_\parallel}{\psi^3} \,,$$

which is the well known PFA for perfect conductors.
NTLO for perfect mirrors ($\lambda \to \infty$) results in:

$$c = \frac{15 - \pi^2}{1080\psi^3} \,,$$

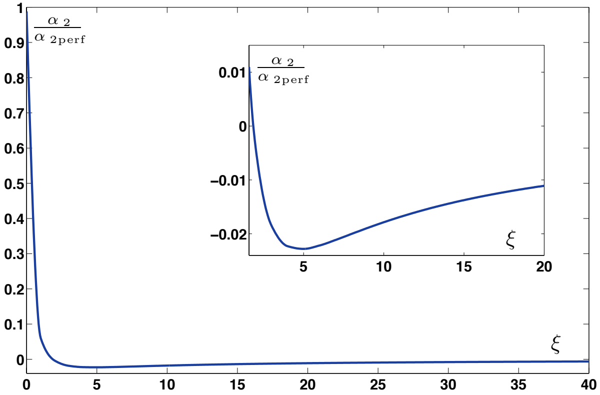
in agreement with the result obtained by Bimonte et al.

# A model

$\lambda(k)$ proportional to $(k_0^2 + \mathbf{k}_\parallel^2)^{-1/2}$. Proportionality constant $\xi$.
By dimensional analysis,

$$E_{vac} \simeq \int d^2\mathbf{x}_\parallel \, \frac{1}{\psi^3} \left[ \alpha_1(\xi_L, \xi_R) + \alpha_2(\xi_L, \xi_R)(\partial_\alpha \psi)^2 \right] .$$

# Ratio between the coefficient of the NTLO correction $\alpha_2(\xi)$ and the corresponding value for the perfect case



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- DE useful in other contexts (we applied it to electrostatics).
- It has been generalized to imperfect mirrors, higher dimensions, and to finite temperature calculations.