Electromagnetic boundary conditions for infinitesimally thin semitransparent δ-function plates

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Abstract

We derive the electromagnetic boundary conditions for infinitesimally thin semitransparent δ-function plates. We show that the optical properties of such a plate are independent of the material properties in the direction normal to the surface. We obtain conditions for the physical realization of such an infinitisimally thin semitransparent δ-function plate. The thin-plate limit of a plasma slab of thickness d with plasma frequency $\omega_p^2 = \zeta_p/d$ reduces to a δ-function plate for frequencies ($\omega = i\zeta$) satisfying $\zeta d \ll \sqrt{\zeta_p d} \ll 1$. We obtain Casimir interaction energy between two delta-finction plates and Casimir Polder energy between an atom and a delta-function plate. In perfect conductor limit these energies give the usual Casimir-Polder energy. The "thick" and "thin" boundary conditions considered by Bordag are found to be identical in the sense that they lead to the same electromagnetic fields.

What is it about?

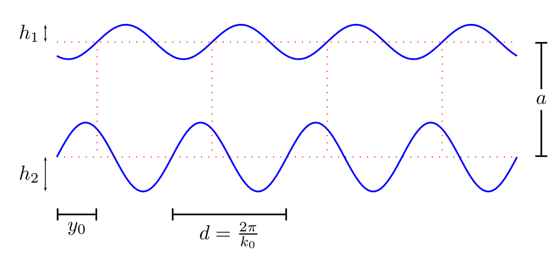
- Electromagnetic boundary conditions on a semitransparent δ-function material.
- What happens to the optical properties of a dielectric slab in zero thickness limit?
- What is the physical realization of a zero thickness material?
- Casimir and Casimir-Polder interaction energy between δ-function materials.

Why are we interested?

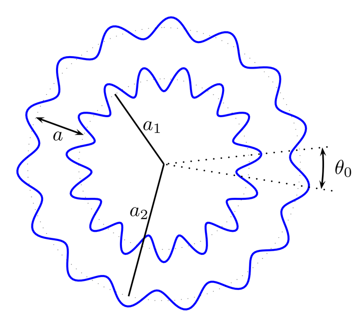
- The Casimir interaction energy between two separate bodies using multiple scattering formalism is

$$E_{12} = \frac{1}{2}\int \frac{d\zeta}{2\pi} \mathrm{Tr} \ln \left[\mathbf{1} - \mathbf{\Gamma}_0 \mathbf{T}_1 \mathbf{\Gamma}_0 \mathbf{T}_2\right], \text{ where } \quad \mathbf{T}_i = \mathbf{V}_i \left[\mathbf{1} - \mathbf{\Gamma}_0 \mathbf{V}_i\right]^{-1}.$$

  It is easier to do analytical calculations if the potentials are described by δ-function.
- Can we describe 2D surfaces like graphene using it?
- Bordag in 2004 showed that Casimir-Polder energy between a δ-function plate and an atom is 13% lower than standard result.
- Earlier work of non-contact gears (PRD 78, 065018, 065019) on scalar field interacting with semitransparent δ-function potential.



Non-contact gears I: Corrugated semi-transparent δ-function plates.



Non-contact gears II: Corrugated semi-transparent δ-functions cylinders.

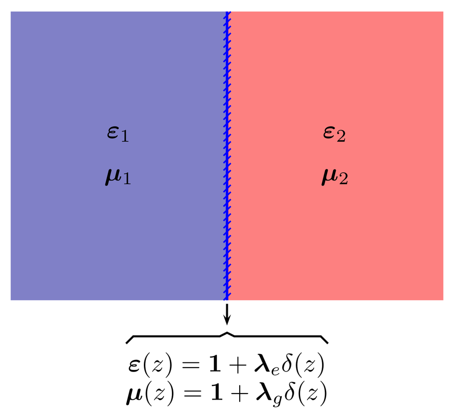
- However, to extend it to the electromagnetic case, we need to answer "what is the physical realization of an infinitesimally thin material?"

Semitransparent δ-function plates

- Electric and magnetic properties:

$$\boldsymbol{\varepsilon}(z) - \mathbf{1} = \boldsymbol{\lambda}_e \delta(z),$$
$$\boldsymbol{\mu}(z) - \mathbf{1} = \boldsymbol{\lambda}_g \delta(z).$$

- Assume that the electric permittivity and the magnetic permeability is isotropic in the plane of the plate only: $\boldsymbol{\varepsilon} = \mathrm{diag}(\varepsilon^{\perp}, \varepsilon^{\perp}, \varepsilon^{||})$ and $\boldsymbol{\mu} = \mathrm{diag}(\mu^{\perp}, \mu^{\perp}, \mu^{||})$.



A semitransparent δ-function plate sandwiched between two semi-infinite slabs.

Fields and Green's functions

- The fields in terms of Green's dyadics are:

$$\mathbf{E}(z) = \int dz' \, \boldsymbol{\gamma}(z,z') \cdot \mathbf{P}(z') \qquad \text{and} \qquad \mathbf{H}(z) = \int dz' \, \boldsymbol{\phi}(z,z') \cdot \mathbf{P}(z').$$

- The reduced electric Green's dyadics is

$$\boldsymbol{\gamma}(z,z') = \begin{bmatrix} \frac{1}{\varepsilon^{\perp}}\frac{\partial}{\partial z}\frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & \frac{1}{\varepsilon^{\perp}}\frac{\partial}{\partial z}\frac{ik_{\perp}}{\varepsilon'^{||}} g^H \\ 0 & \omega^2 g^E & 0 \\ -\frac{ik_{\perp}}{\varepsilon^{||}}\frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & -\frac{ik_{\perp}}{\varepsilon^{||}}\frac{ik_{\perp}}{\varepsilon'^{||}} g^H \end{bmatrix}$$

- The reduced magnetic Green's dyadics is

$$\boldsymbol{\phi}(z,z') = i\omega \begin{bmatrix} 0 & \frac{1}{\mu^{\perp}}\frac{\partial}{\partial z} g^E & 0 \\ \frac{1}{\varepsilon'^{\perp}}\frac{\partial}{\partial z'} g^H & 0 & \frac{ik_{\perp}}{\varepsilon'^{||}} g^H \\ 0 & -\frac{ik_{\perp}}{\mu^{||}} g^E & 0 \end{bmatrix}.$$

- Magnetic Green's function $g^H(z,z')$ obeys

$$\left[-\frac{\partial}{\partial z}\frac{1}{\varepsilon^{\perp}}\frac{\partial}{\partial z} + \frac{k_{\perp}^2}{\varepsilon^{||}} - \omega^2 \mu^{\perp}\right] g^H(z,z') = \delta(z-z')$$

- Electric Green's function $g^E(z,z')$ obeys

$$\left[-\frac{\partial}{\partial z}\frac{1}{\mu^{\perp}}\frac{\partial}{\partial z} + \frac{k_{\perp}^2}{\mu^{||}} - \omega^2 \varepsilon^{\perp}\right] g^E(z,z') = \delta(z-z')$$

Boundary conditions on scalar Green's functions

- TM mode

$$g^H(z,z')\Big|_{a-\delta}^{a+\delta} = \frac{\lambda_e^{\perp}}{2}\left[\left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H\right\}_{a+\delta} + \left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H\right\}_{a-\delta}\right]$$
$$\left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H(z,z')\right\}\Bigg|_{a-\delta}^{a+\delta} = \zeta^2 \frac{\lambda_g^{\perp}}{2}\left[g^H(a+\delta,z') + g^H(a-\delta,z')\right]$$

- TE mode

$$g^H(z,z')\Big|_{a-\delta}^{a+\delta} = \frac{\lambda_e^{\perp}}{2}\left[\left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H\right\}_{a+\delta} + \left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H\right\}_{a-\delta}\right]$$
$$\left\{\frac{1}{\varepsilon^{\perp}(z)}\frac{\partial}{\partial z} g^H(z,z')\right\}\Bigg|_{a-\delta}^{a+\delta} = \zeta^2 \frac{\lambda_g^{\perp}}{2}\left[g^H(a+\delta,z') + g^H(a-\delta,z')\right]$$

Reflection and transmission coefficients

- The reflection coefficients

$$r_{ij}^H = \frac{\bar{\kappa}_i^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_j^H}{2}\right)\left(1-\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_i^H}\right) - \bar{\kappa}_j^H\left(1-\frac{\lambda_e^{\perp}\bar{\kappa}_i^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_j^H}\right)}{\bar{\kappa}_i^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_j^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_i^H}\right) + \bar{\kappa}_j^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_i^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_j^H}\right)}$$

- The transmission coefficients

$$t_{ij}^H = \frac{\bar{\kappa}_i^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_j^H}{2}\right)\left(1-\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_i^H}\right) + \bar{\kappa}_i^H\left(1-\frac{\lambda_e^{\perp}\bar{\kappa}_i^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_i^H}\right)}{\bar{\kappa}_i^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_j^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_i^H}\right) + \bar{\kappa}_j^H\left(1+\frac{\lambda_e^{\perp}\bar{\kappa}_i^H}{2}\right)\left(1+\frac{\lambda_g^{\perp}\zeta^2}{2\bar{\kappa}_j^H}\right)}$$

- $\kappa_i^H = \sqrt{k_{\perp}^2 \frac{\varepsilon_i^{\perp}}{\varepsilon_i^{||}} + \zeta^2 \varepsilon_i^{\perp}\mu_i^{\perp}}$ and $\bar{\kappa}_i^H = \frac{\kappa_i^H}{\varepsilon_i^{\perp}} = \sqrt{\frac{k_{\perp}^2}{\varepsilon_i^{\perp}\varepsilon_i^{||}} + \zeta^2 \frac{\mu_i^{\perp}}{\varepsilon_i^{\perp}}}.$
- The electric coefficients are obtained by replacing $\boldsymbol{\varepsilon} \leftrightarrow \boldsymbol{\mu}$ and $H \to E$.

Green's functions for a semitransparent δ-function plate

- The magnetic Green's function is

$$g^H(z,z') = \frac{1}{2\kappa} e^{-\kappa|z-z'|} + \left[r_g^H + \eta(z-a)\eta(z'-a)\, r_e^H\right] \frac{1}{2\kappa} e^{-\kappa|z-a|} e^{-\kappa|z'-a|},$$

  where we have set $\varepsilon_i^{\perp} = \varepsilon_i^{||} = 1$ and $\mu_i^{\perp} = \mu_i^{||} = 1$. $\eta(z)$ is the sign-function and $\kappa = \sqrt{k_{\perp}^2 + \zeta^2}$.
- The reflection and the transmission coefficients

$$r_e^H = \frac{\lambda_e^{\perp}}{\lambda_e^{\perp} + \frac{2}{\kappa}}, \qquad \text{and} \qquad r_g^H = -\frac{\lambda_g^{\perp}}{\lambda_g^{\perp} + \frac{2\kappa}{\zeta^2}}$$

A perfect electric and magnetic conductor will be transparent ($t^H = -1$). A circular laser, of circumference equal to its' wavelength, would extinguish itself going through such a plate.

Physical realization of a semitransparent δ-function plate

- Consider a slab of thickness d. Limit $d \to 0$ gives the δ-function response of dielectric permittivity.

$$\varepsilon^{\perp}(i\zeta) - 1 = \lambda_e^{\perp}(i\zeta) \lim_{d \to 0} \frac{[\theta(z+d) - \theta(z)]}{d},$$

- In thin-plate limit: $\zeta^2 d^2 \ll \frac{d}{\lambda_e^{\perp}} \ll 1$ and $k_{\perp}^2 d^2 \ll \frac{d}{\lambda_e^{\perp}} \ll 1$, the refection and transmission coefficient for a slab reproduces the respective coefficients for the δ-function plate.

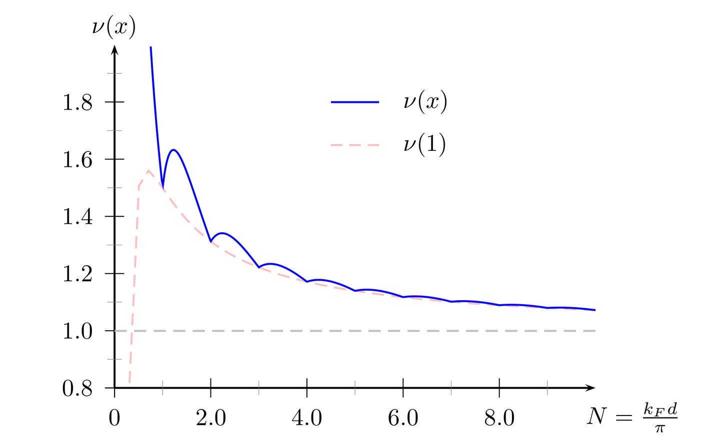
$$r_{\text{thick}}^H = -\left(\frac{\bar{\kappa}_i^H - \kappa}{\bar{\kappa}_i^H + \kappa}\right) \frac{(1 - e^{-2\kappa_i^H d})}{\left[1 - \left(\frac{\bar{\kappa}_i^H - \kappa}{\bar{\kappa}_i^H + \kappa}\right)^2 e^{-2\kappa_i^H d}\right]} \to r_e^H = \frac{\lambda_e^{\perp}}{\lambda_e^{\perp} + \frac{2}{\kappa}},$$

$$r_{\text{thick}}^E = -\left(\frac{\kappa_i^E - \kappa}{\kappa_i^E + \kappa}\right) \frac{(1 - e^{-2\kappa_i^E d})}{\left[1 - \left(\frac{\kappa_i^E - \kappa}{\kappa_i^E + \kappa}\right)^2 e^{-2\kappa_i^E d}\right]} \to r_e^E = -\frac{\lambda_e^{\perp}}{\lambda_e^{\perp} + \frac{2\kappa}{\zeta^2}}$$

Plasma sheet

- Plasma model realization requires identifying $\lambda_e^{\perp} = \frac{\zeta_p}{\zeta^2}$, where $\zeta_p = \frac{e^2}{m}\frac{n_{\text{tot}}}{A} \to \omega_p^2 d$.
- This suggests that number density n_f is inversely proportional to the thickness of the material for a very thin plasma sheet.
- We model charge carriers as a non-relativistic Fermi gas confined in a thin material slab of thickness d with no flux accross the walls.
- Number density in terms of Fermi wave-vector k_F^2 is given by $n_f = \frac{k_F^3}{3\pi^2}\nu(x)$, where A is the area of the slab and the thickness dependence is given by function $\nu(x)$

$$\nu(x) = \frac{3}{2}\left(x - \frac{1}{3}x^3\right) + \frac{3}{2N}\left(1 - \frac{1}{2}x^2\right) - \frac{1}{4N^2}x, \quad \text{where} \quad x = \frac{[N]}{N} \quad \text{and} \quad N = \frac{k_F d}{\pi}.$$



Plot of $\nu(x)$ in Eq. (3) versus $N = k_F d/\pi$. The value of $\nu(1)$ and of $N \to \infty$ limit are also shown.

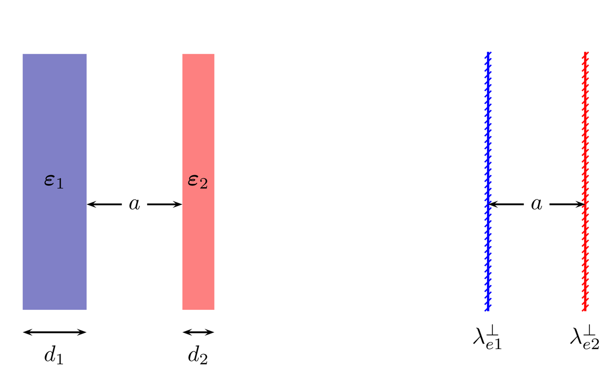
The oscillations in $\nu(x)$ resembles the de Hass–van Alphen effect resulting due to the quantization effect due to presence of the magnetic field.

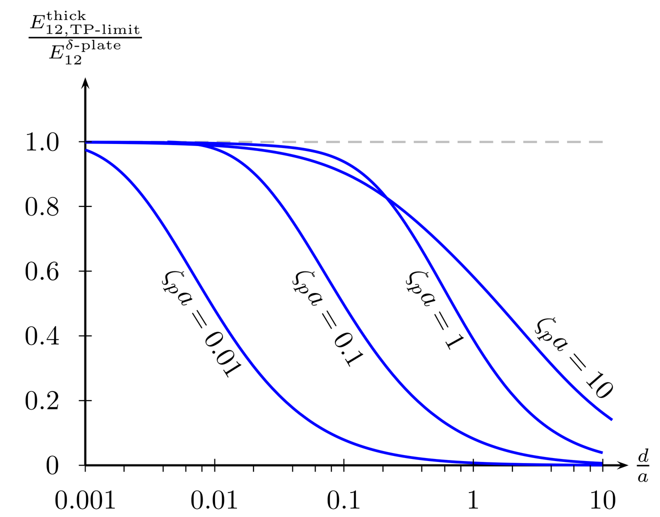
Casimir energy for δ-function plates

- Two-body contribution to energy: $E_{\text{tot}} = E_0 + E_1 + E_2 + E_{12}$.

$$\frac{E_{12}}{A} = \frac{1}{2}\int \frac{d\zeta \, d^2k}{(2\pi)^3}\left[\ln\left\{1 - r_1^E r_2^E e^{-2\kappa a}\right\} + \ln\left\{1 - r_1^H r_2^H e^{-2\kappa a}\right\}\right],$$

  where r_i^E and r_i^H are reflection coefficients of individual objects.



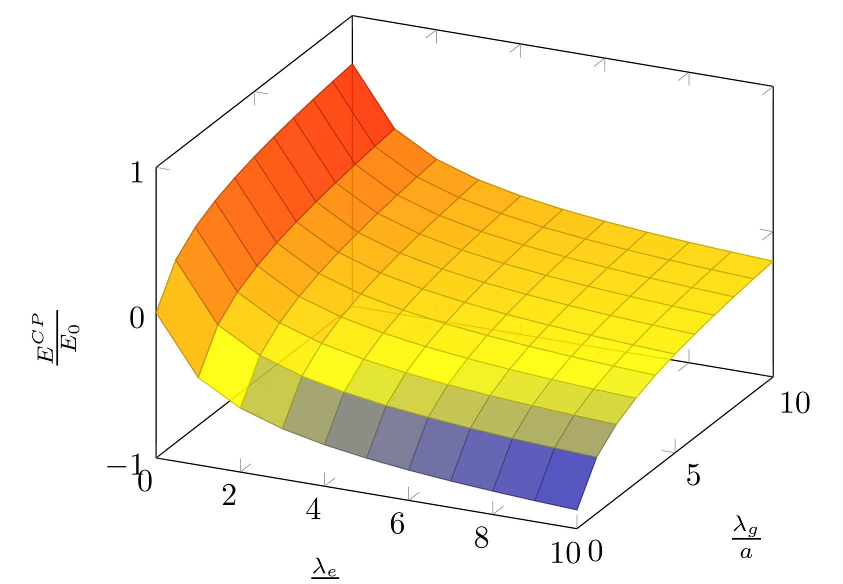


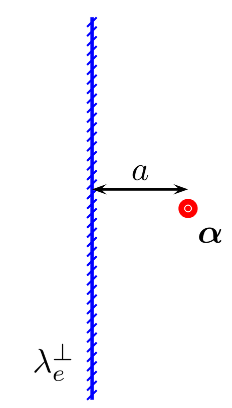
- Thin-plate limit: $\frac{d}{a} \ll \zeta_p a \ll \frac{1}{d/a}$.

Casimir-Polder energy

- Atom and a δ-function plate:

$$E_{12}^{\text{CP}} = -2\pi\alpha \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \int \frac{d^2k_{\perp}}{(2\pi)^2} \frac{e^{-2\kappa a}}{2\kappa}\left[2k_{\perp}^2 r^H + \zeta^2(r^H - r^E)\right].$$





- Casimir-Polder energy between atom and a perfect conductor is $E_0 = -\frac{3\alpha}{8\pi a^4}$.
- We do not see difference in the "thin" boundary condition described by Bordag. The "thick" and "thin" propagators described in his paper corresponds to the same Green's dyadic and therefore corresponds to same physical situation.

References:

[1] P. Parashar, K. A. Milton, K. V. Shajesh and M. Schaden, "Electromagnetic semitransparent δ-function plate: Casimir interaction energy between parallel infinitesimally thin plates," arXiv:1206.0275 [hep-th].(To be published in PRD).