1. (15) Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{R}$ which are similar over $\mathbb{C}$. Prove or disprove that $A$ is similar to $B$ over $\mathbb{R}$.
2. (15) Let $A$ be an $n \times n$ matrix with entries in $\mathbb{C}$ such that for each eigenvalue $c$ of $A$, the eigensubspace of $A$ associated with $c$ is one dimensional. Let $B$ be any $n \times n$ matrix satisfying $AB = BA$. Show that there is a polynomial $f \in \mathbb{C}[x]$ such that $B = f(A)$. 
3. (15) Let $V$ be a finite dimensional vector space over the field $F$ and $T$ be a linear transformation on $V$. For any vector $\alpha \in V$, let $Z(\alpha; T) = \{ f(T)\alpha \mid f \in F[x]\}$. Suppose that $V$ has two cyclic decompositions:

- $V = Z(\beta_1; T) \oplus Z(\beta_2; T)$, with the $T$-annihilator of $\beta_i$ being $q_i$ and satisfying $q_1$ divides $q_2$;
- $V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$, with the $T$-annihilator of $\alpha_i$ being $p_i$ satisfying $p_i$ divides $p_{i+1}$ for $i = 1, 2, \ldots, r - 1$.

Show that $r = 2$, $p_1 = q_1$ and $p_2 = q_2$. 

4. (30) Let \( A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 35 & 21 & 1 & 0 \\ -15 & -9 & 0 & 1 \end{pmatrix} \).

a) Find the invariant factors, minimal polynomial and the characteristic polynomial of \( A \).

b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
5. (15) Let $T$ be a linear operator on the $n$-dimensional inner product space $V$ over $\mathbb{C}$ satisfying the property that $TT^* = f(T)$, where $f$ is any polynomial in $\mathbb{C}[x]$ such that $f(0) = 0$. Show that $T$ is normal. (Recall that the adjoint operator $T^*$ is defined by the equality $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all vectors $\alpha$ and $\beta$ in $V$ and $T$ is normal if $TT^* = T^*T$.)