1. Solve the following initial value problem.

\[ z^2 z_x + x^3 y z_y = x^3 z, \]

with \( z = y^2 \) on the line \( x = 2y, \ y > 0 \).

2. Find the solution to the Dirichlet problem in the annulus:

\[ \Delta u = 0, \quad \frac{1}{2} < r < 2, \quad -\pi \leq \theta \leq \pi, \]

\[ u \left( \frac{1}{2}, \theta \right) = u \left( 2, \theta \right) = \cos^3 \theta, \quad -\pi \leq \theta \leq \pi. \]

3. Let \( \Omega \) be a bounded normal domain in \( \mathbb{R}^3 \). Suppose that \( h(x) \) is harmonic in \( \Omega \) and that

\[ h(x) = \frac{1}{4\pi ||x-y||}, \quad \text{if} \quad x \in \partial \Omega, \]

where \( y \in \Omega \). Let

\[ g(x) = -\frac{1}{4\pi ||x-y||} + h(x). \]

Evaluate

\[ \int_{\partial \Omega} x_1 \frac{\partial g(x)}{\partial n} d\sigma(x). \]

4. Let \( \Omega = (-R, R) \times ... \times (-R, R) \subset \mathbb{R}^n \) be the cube in \( \mathbb{R}^n \) centered at the origin and with side-length \( 2R \). Suppose that \( u \in C^2 \cap C(\bar{\Omega}) \) be a solution to the problem

\[ \Delta u = -1 \quad \text{in} \quad \Omega, \]

\[ u = 0, \quad \text{on} \quad \partial \Omega. \]

Prove that \( u \) satisfy the estimate

\[ \frac{R^2}{2n} \leq u(0) \leq \frac{R^2}{2}. \]
5. Find the solution of the following initial-boundary value problem for
the wave equation in $\mathbb{R}^2$

\[
\begin{aligned}
\Delta u - u_{tt} &= 0, \quad (x,y) \in \Omega, \quad t > 0 \\
u|_{t=0} &= 3 \sin x \sin 2y \\
u_t|_{t=0} &= 5 \sin 3x \sin 4y \\
u|_{\partial \Omega} &= 0
\end{aligned}
\]

where $\Omega$ is the rectangle,

\[
\Omega = \{(x,y) : 0 < x < \pi, \quad 0 < y < \pi\}.
\]

6. Suppose a $C^2$ solution exists to

\[
\begin{aligned}
u_t &= \Delta u, \quad x \in \Omega, \quad t > 0, \\
u &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x,0) &= \phi(x), \quad x \in \Omega,
\end{aligned}
\]

for a normal bounded domain $\Omega \subset \mathbb{R}^n$. Define $F$ by

\[
F(t) = \int_\Omega u^2 \, dx.
\]

(a) Show that $\frac{dF}{dt} \leq 0$ and hence $0 \leq F(t) \leq F(0)$.

(b) Use the part (a) to show that there is at most one $C^2$ solution to
the initial-boundary problem.