Problem 1. The Riemann function $R(x)$ on the segment $[0, 1]$ is defined as

$$R(x) = \begin{cases} 
0, & \text{when } x \text{ is irrational or } x = 0, \\
\frac{1}{q}, & \text{when } x = \frac{p}{q} > 0, \text{ where } p \text{ and } q \text{ are relatively prime.}
\end{cases}$$

Is there a sequence $\langle f_n \rangle$ of continuous functions on $[0, 1]$ which converges to $R(x)$ at every point $x \in [0, 1]$? Explain.

Problem 2. Let us enumerate all rational points on the segment $[0, 1]$, $\mathbb{Q} \cap [0, 1] = \{ r_1, r_2, \ldots \}$.

Define the function

$$f(x) = \sum_{n: \ r_n < x} \frac{1}{2^n}.$$ 

Prove that

1. the function $f(x)$ is discontinuous at every rational point,
2. the function $f(x)$ is continuous at every irrational point,
3. the function $f(x)$ is differentiable almost everywhere,
4. $f'(x) = 0$ almost everywhere.

Problem 3. Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^{\infty} E_n$ has a finite Lebesgue measure. Prove that for any set $A \subset \mathbb{R}$,

$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).$$ 

Problem 4. Let $f \in L^1(-\infty, \infty)$. Find the limit,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{f(x)e^{nx}dx}{1 + e^{nx}},$$

and justify your answer.

Problem 5. Consider the functional

$$F(f) = \int_0^1 e^x f(x)dx.$$ 

Prove that $F$ is a linear bounded functional on $L^4[0, 1]$ and calculate the norm of $F$ on $L^4[0, 1]$. 
Problem 6. Prove that if \( f, g \in L^1(-\infty, \infty) \) and

\[
    h(x) = \int_{-\infty}^{\infty} f(x - e^y)g(y)dy,
\]

then \( h \in L^1(-\infty, \infty) \) and

\[
    \|h\|_1 \leq \|f\|_1 \|g\|_1.
\]