Problem 1. Prove that the space $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$, with the norm,

$$\|f(x)\|_{C^1[0, 1]} = \sup_{0 \leq x \leq 1} (|f(x)| + |f'(x)|),$$

is separable.

Problem 2. Find a linear function $ax + b$ such that the norm,

$$\|ax + b - e^x\|_{L^2[0, 1]},$$

is minimal.

Problem 3. Let $f(x)$ be a continuous function on $\mathbb{R}$, and let $g(x)$ be a measurable function on $[0, 1]$. Is the function $f(g(x))$ measurable? Explain.

Problem 4. Is there an integrable function $f(x)$ on $[0, 1]$ such that for any $0 < a < b < 1$,

$$\|f(x)\|_{L^2[a, b]} = \infty.$$

Explain.

Problem 5. Is there an open connected set in $\mathbb{R}^2$ of finite Lebesgue measure, which has a non-empty intersection with any straight line? Explain.

Problem 6. Prove that there is a constant $C > 0$ such that for any continuous function $f$ on $[0, 1]$,

$$\|\ln(1 + |f|)\|_{L^1[0, 1]} \leq C\|f\|_{L^2[0, 1]}.$$