Problem 1. Let $G$ be a cyclic group of order 2008 with generator $a$. Find all $b \in G$ such that $G/\langle b \rangle$ is isomorphic to $\langle a^{100} \rangle$. (Here $\langle c \rangle$ denotes the subgroup of $G$ generated by $c \in G$.)

Problem 2. Let $G$ be a group of order $2^2 \cdot 3 \cdot 11$. Show that $G$ is not simple.

Problem 3. Let $G = SL(2, \mathbb{Z})$ be the group of $2 \times 2$ matrices with integer entries and determinant 1. Let $H \subset \mathbb{C}$ be the upper half plane of complex numbers, $H = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$. Show that the formula
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}
\]
defines an action of $G$ on $H$ and show that all orbits are infinite.

Problem 4. Let $R$ be a commutative ring with ideals $I$ and $J$. Assume that $IJ$ is a principal ideal. Show that both $I$ and $J$ are finitely generated.

Problem 5. Show that the ring $R = \{ a + b\sqrt{-2}, \ a, b \in \mathbb{Z} \}$ is an Euclidean domain with the norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$ and therefore a unique factorization domain. Write the prime decomposition of 6 in $R$.

Problem 6. Compute the Galois group of the polynomial $p(x) = x^4 - 14x^2 + 9$ over $\mathbb{Q}$. List all the subgroups of the Galois group and the corresponding intermediate fields.

Problem 7. Describe explicitly an injective homomorphism from the symmetric group $S_3$ to the group $GL(2, \mathbb{R})$ of $2 \times 2$ invertible matrices with real coefficients. (Hint: $S_3$ acts in $\mathbb{R}^3$ by permuting the coordinates of the vectors and the 2-dimensional plane $x + y + z = 0$ is invariant under this action.)

Problem 8. Let $G = \mathbb{Z} + \mathbb{Z}$ be a free abelian group of rank 2 with generators $x$ and $y$. Let $H \subset G$ be the subgroup generated by two elements $2x + 2y$ and $x + 3y$. Describe the groups $H$ and $G/H$. 